Approximate Ramsey theory

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Ultrafilters, Ramsey theory and dynamics
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November 23, 2017
Ramsey’s theorem

For every $k \leq m$, $r \geq 2$, and every colouring of $k$-element subsets of $\mathbb{N}$ with $r$-many colours there is an infinite subset $X$ of $\mathbb{N}$ such that all $k$-element subsets of $X$ have the same colour.
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Finite Ramsey’s theorem

For every $k \leq m$ and $r \geq 2$, there exists $n$ such that for every colouring of $k$-element subsets of $n$ with $r$-many colours there is a subset $X$ of $n$ of size $m$ such that all $k$-element subsets of $X$ have the same colour.
THE CLASS OF FINITE LINEAR ORDERS IS RAMSEY

Given $A$ and $B$ finite linear orders, $|A| \leq |B|$ and $r \geq 2$, there exists a finite linear order $C$ such that whenever we colour copies of $A$ in $C$ by $r$ colours, there is a copy $B'$ of $B$ in $C$ such that all copies of $A$ in $B'$ have the same colour.
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A topological group $G$ is extremely amenable if it has a fixed point under any continuous action on a compact Hausdorff space. Equivalently, every minimal $G$-flow is a singleton.
Proof

Topology on \( G = \text{Aut}(\mathbb{Q}, <) \) is given by stabilizers of finite suborders

\[ G_A = \{ g \in G : ga = a \ \forall a \in A \} \]
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Finite linear orders are a Ramsey class $\longleftrightarrow$ every partition

$G = \bigcup_{i=1}^r G_A K_i$ has a thick part $\longleftrightarrow$ there are no disjoint topologically syndetic sets.
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**Theorem (Kechris, Pestov, and Todorčević)**

$G$ is extremely amenable iff finitely generated substructures of $\mathcal{A}$ form a rigid Ramsey class.
Examples

RAMSEY CLASSES

- finite linear orders (Ramsey);
- finite linearly ordered graphs (Nešetřil and Rödl);
- finite linearly ordered metric spaces (Nešetřil);
- finite Boolean algebras (Graham and Rothschild).
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**EXTREMELY AMENABLE GROUPS**
1. \(\text{Aut}(\mathbb{Q}, <)\) (Pestov);
2. \(\text{Aut}(\mathcal{R}, <)\) = group of automorphisms of the random ordered graph (KPT);
3. \(\text{Iso}(\mathbb{U}, d)\) = group of isometries of the Urysohn space (Pestov);
4. \(U(l_2)\) = group of unitaries of the separable Hilbert space (Gromov + Milman);
5. \(\text{LIso}(\mathcal{G})\) = group of linear isometries of the Gurarij space (B + López-Abad + Mbombo).
Theorem (Melleray-Tsankov)

*For M approximately ultrahomogeneous, Iso(M) is extremely amenable $\iff$ finitely-generated substructures satisfy the approximate Ramsey property (ARP).*
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FIRST EXAMPLE (Pestov)
$\text{Iso}(U,d)$ is e.a. $\iff$ finite metric spaces satisfy ARP.

PREFIRST EXAMPLE (Gromov + Milman)
$U(\ell^2)$ is e.a. $\iff$ finite dimensional inner spaces satisfy ARP.

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$\text{Iso}(l(G))$ is e.a. $\iff$ finite dimensional Banach spaces satisfy ARP.
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Kubiś-Solecki; Henson

Simple proof - metric Fraïssé theory.

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(3) for every $E$ finite dimensional, $i : E \hookrightarrow G$ isometric embedding and $\varepsilon > 0$ there is a linear isometry $f : G \rightarrow G$

$$\|i - f \upharpoonright E\| < \varepsilon$$
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Katětov construction
Approximate Ramsey property for $l^n_{\infty}$'s

Theorem (B+LA+M)

Let $d$ be the number of colours $\varepsilon > 0$.

Then for every colouring $c : \text{Emb}(l^d_{\infty}, l^n_{\infty}) \to \{0, 1, \ldots, r - 1\}$, there is an $i \in \text{Emb}(l^d_{\infty}, l^n_{\infty})$ and $\alpha < r$ such that $i \circ \text{Emb}(l^d_{\infty}, l^n_{\infty}) \subset (c - 1)(\alpha)$.

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$\text{Iso}(G)$ is extremely amenable.
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**Theorem (B+LA+M)**

\( \text{Iso}_l(\mathbb{G}) \) \textit{is extremely amenable.}
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About the proof

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**GENERAL CASE**
Discretize and use dual Ramsey theorem of Graham and Rothschild.
Further structures

$P$ – Poulsen simplex

$p$ – extreme point in $P$

$\text{AH}(P, p)$ – group of affine homeomorphisms of $P$ fixing $p$
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$\text{AH}(P, p)$ is e.a. $\iff$ finite dimensional simplexes with a distinguished extreme point satisfy ARP

Continuum is a compact connected space.

$L$ – Lelek fan ($\equiv$ unique subcontinuum of the Cantor fan with dense set of endpoints)

$<$ – total order on endpoints of type ($\mathbb{Q}, <$)

**Theorem (B+Kwiatkowska)**

$\text{Homeo}(L, <)$ is e.a. $\iff$ generalization of Gowers’ Hindman’s theorem.
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Big problem

\[
P - \text{pseudoarc}
\]

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p \in P
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Big problem

$P$ – pseudoarc

$p \in P$

Is $\text{Homeo}(P, p)$ e.a.?
THANK YOU!