Results in Density Ramsey Theory.

Konstantinos Tyros

Lyon, 2017

Szeméredi's Theorem

Theorem (Szeméredi, 1975)

Let k be a positive integer and δ be a positive real. Then there exists some positive integer n_0 such that for every $n \geqslant n_0$ we have that every subset A of the set $\{1, ..., n\}$ of cardinality at least δn contains an arithmetic progression of length k.

The Hales-Jewett Theorem

To state the Hales-Jewett Theorem we need to introduce some notation. Let k be a positive integer and n be a non-negative integer.

- By $[k]^n$ we denote the set of all sequences $(a_0, ..., a_{n-1})$ of length n with elements from $[k]$.
- We will refer to the elements of $[k]^n$ as constant words of length n over k.

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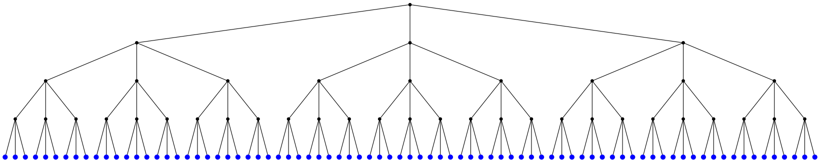
The set $[3]^4$ is



Moreover, let v be a symbol not belonging to $[k] = \{1, ..., k\}$.

- A variable word over k is a finite sequence $w(v)$ in $[k] \cup \{v\}$ such that v occurs at least once.
- For a variable word $w(v)$ and $a \in [k]$ by $w(a)$ we denote the constant word over k resulting by substituting every occurrence of v by a.
- A combinatorial line is a set of the form $\{w(a) : a \in [k]\}$, where $w(v)$ is a variable word over k.
- Given a variable word $w(v) = (\alpha_0, ..., \alpha_{n-1})$, the set $\{i \in \{0, ..., n-1\} : w_i = v\}$ is called the wildcard set of $w(v)$.

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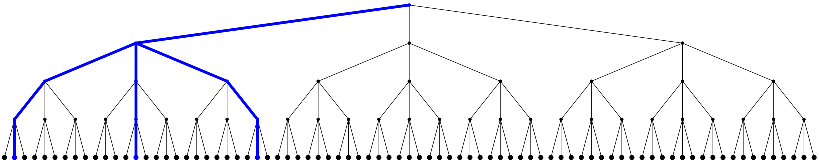
For example assume that $k = 3$ and $n = 4$. Also let $w(v) = (1, v, v, 2)$. Then the corresponding combinatorial line is the set

$$\{(1, \mathbf{1}, \mathbf{1}, 2), (1, \mathbf{2}, \mathbf{2}, 2), (1, \mathbf{3}, \mathbf{3}, 2)\}$$



The wildcard set is $\{1, 2\}$.

The Hales-Jewett Theorem

Theorem (Hales and Jewett, 1963)

Let k, r be positive integers. Then there exists an integer n_0 such that for every $n \geqslant n_0$ and every r-coloring of $[k]^n$ there exists a variable word $w(v)$ of length n such that the set $\{w(a) : a \in [k]\}$ is monochromatic.

- The least such n_0 is denoted by $HJ(k, r)$.
- The best known upper bounds for the numbers $HJ(k, r)$ are primitive recursive and are due to Shelah (1988).

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The Graham–Rothschild Theorem

An m-variable word $w(v_0, ..., v_{m-1})$ is a sequence in $[k] \cup \{v_0, ..., v_{m-1}\}$ such that each v_i appears at least once and their appearances are in block position.
An m-dimensional subspace of $[k]^n$ is a subset of $[k]^n$ of the form $\{w(a_0, ..., a_{m-1}) : a_0, ..., a_{m-1} \in [k]\}$, where $w(v_0, ..., v_{m-1})$ is an m-variable word of length n.

Theorem (Graham–Rothschild Theorem)

Let k, d, m, r be positive integers with $d \leqslant m$. Then there exists a positive integer n_0 such that for every $n \geqslant n_0$ and every r-coloring of the d-dimensional subspaces of $[k]^n$ there exists an m-dimensional subspace S of $[k]^n$ such that the set of d-dimensional subspaces of S is monochromatic.

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The Density Hales-Jewett Theorem

Theorem (Furstenberg and Katznelson, 1991)

Let k be positive integer and δ be a real with $0 < \delta \leqslant 1$. Then there exists an integer n_0 such that for every $n \geqslant n_0$ and every subset A of $[k]^n$ of uniform density at least δ, that is $\frac{|A|}{k^n} \geqslant \delta$, there exists a combinatorial line contained in A.

- The least such n_0 is denoted by $DHJ(k, \delta)$.
- Furstenberg and Katznelson proved it using Ergodic Theory.
- In 2011, Austin gave another proof using Ergodic theoretic techniques.
- A combinatorial proof is provided by the Polymath paper (2012), giving upper bounds for the numbers $DHJ(k, \delta)$ which have an Ackermann type dependence on k.
- Tao presents another proof on his blog.
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Let k, m be a positive integers.

- By $[k]^{<\omega}$ we denote the set of all sequences.
- A Carlson-Simpson tree of dimension m is a set of the form

$$\{c\} \cup \{c^\frown w_0(a_0)^\frown \ldots ^\frown w_n(a_n) : n \in \{0, \ldots, m-1\} \text{ and } a_0, \ldots, a_n \in [k]\}$$

where c is a constant word over k and $w_0(v), \ldots, w_{m-1}(v)$ are left variable words.

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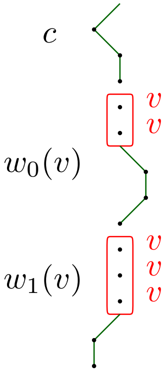

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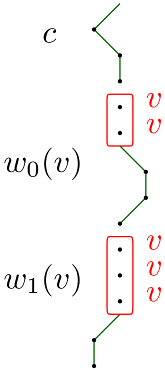

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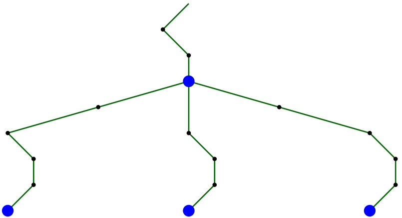

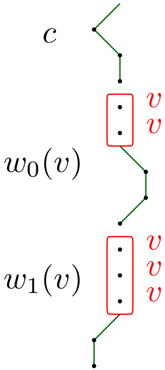

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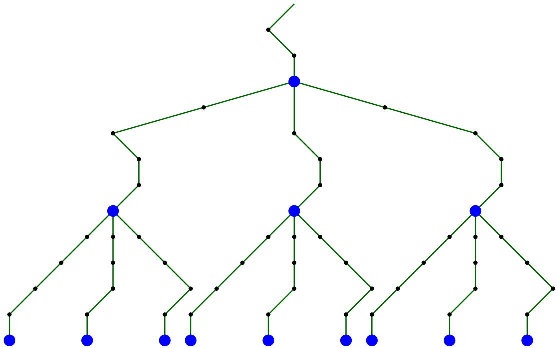

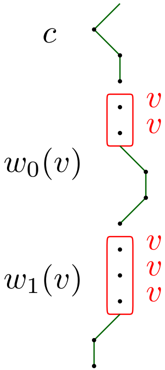

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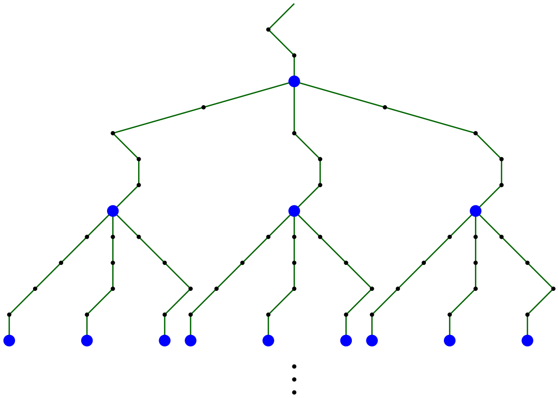

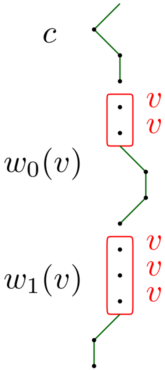

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Let k be a positive integer. Then every subset A of $[k]^{<\omega}$ satisfying

$$\limsup_{n\to\infty} \frac{|A \cap [k]^n|}{|[k]^n|} > 0$$

contains an infinite Carlson-Simpson tree.

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Let k, m be a positive integer and δ be a real with $0 < \delta \leqslant 1$. Then there exists an integer n_0 with the following property. If A is a subset of $[k]^{<\omega}$ such that for at least n_0 many n's

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then A contains a Carson–Simpson tree of dimension m.

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Theorem

Let k be positive integer and δ be a real with $0 < \delta \leqslant 1$. Then there exists an integer $n = n(k, \delta)$ and a positive real $\theta = \theta(k, \delta)$ with the following property. For every family of measurable events $(A_t)_{t \in [k]^n}$ in a probability space (Ω, Σ, μ) such that $\mu(A_t) \geqslant \delta$ for all $t \in [k]^n$, there exists a variable word $w(v)$ of length n such that

$$\mu\Big(\bigcap_{a \in [k]} A_{w(a)}\Big) \geqslant \theta$$

A regularity technique

Let I be a non-empty interval of $\{0, ..., n-1\}$. We set $I^c = \{0, ..., n-1\} \setminus I$. By $[k]^I$ (resp. $[k]^{I^c}$) we denote the set of all maps from I (resp. $[k]^{I^c}$) into $[k]$. Then $[k]^n \equiv [k]^I \times [k]^{I^c}$ sending each pair (x, y) to $x \cup y$.

If A is a subset of $[k]^n$ and $x \in [k]^I$, we set

$$A_x = \{y \in [k]^{I^c} : x \cup y \in A\}.$$

Lemma

Let k, m be positive integers and ε a positive real. Then there exists an n_0 such that for every $n \geqslant n_0$ and every subset A of $[k]^n$, there exists an interval I of $\{0, ..., n-1\}$ of length m such that

$$|\mathrm{dens}(A) - \mathrm{dens}(A_x)| < \varepsilon$$

for all x in $[k]^I$.

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A regularity technique

Let I be a non-empty interval of $\{0, ..., n-1\}$. We set $I^c = \{0, ..., n-1\} \setminus I$. By $[k]^I$ (resp. $[k]^{I^c}$) we denote the set of all maps from I (resp. $[k]^{I^c}$) into $[k]$. Then $[k]^n \equiv [k]^I \times [k]^{I^c}$ sending each pair (x, y) to $x \cup y$.

If A is a subset of $[k]^n$ and $x \in [k]^I$, we set

$$A_x = \{y \in [k]^{I^c} : x \cup y \in A\}.$$

Lemma

Let k, m be positive integers and ε a positive real. Then there exists an n_0 such that for every $n \geqslant n_0$ and every subset A of $[k]^n$, there exists an interval I of $\{0, ..., n-1\}$ of length m such that

$$|\text{dens}(A) - \text{dens}(A_x)| < \varepsilon$$

for all x in $[k]^I$.

A concentration inequality

Let n be a positive integer and let $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1), \dots, (\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$ be standard Borel probability spaces. By $(\mathbf{\Omega}, \boldsymbol{\mathcal{F}}, \mathbf{P})$ we denote their product. More generally, for every nonempty subset I of $\{1, \dots, n\}$ by $(\mathbf{\Omega}_I, \boldsymbol{\mathcal{F}}_I, \mathbf{P}_I)$ we denote the product of the spaces $\{(\Omega_i, \mathcal{F}_i, \mathbb{P}_i) : i \in I\}$. In particular, we have

$$\mathbf{\Omega} = \prod_{i=1}^{n} \Omega_i \text{ and } \mathbf{\Omega}_I = \prod_{i \in I} \Omega_i.$$

Now let $f \colon \mathbf{\Omega} \to \mathbb{R}$ and $I \subseteq \{1, \dots, n\}$ such that I and I^c are nonempty. For every $\mathbf{x} \in \mathbf{\Omega}_I$ let $f_{\mathbf{x}} \colon \mathbf{\Omega}_{I^c} \to \mathbb{R}$ be defined by the rule $f_{\mathbf{x}}(\mathbf{y}) = f\big(\mathbf{x} \cup \mathbf{y}\big)$.

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A concentration inequality

Theorem (Dodos, Kanellopoulos and T.)

Let $0 < \varepsilon \leqslant 1$ *and* $1 < p \leqslant 2$ *and set*

$$c(\varepsilon, p) = \frac{1}{4}\varepsilon^{\frac{2(p+1)}{p}}(p-1).$$

Also let n *be a positive integer with* $n \geqslant c(\varepsilon, p)^{-1}$ *and let* $(\boldsymbol{\Omega}, \boldsymbol{\mathcal{F}}, \mathbf{P})$ *be the product of the probability spaces* $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1), \ldots, (\Omega_n, \mathcal{F}_n, \mathbb{P}_n)$. *Then for every* $f \in L_p(\boldsymbol{\Omega}, \boldsymbol{\mathcal{F}}, \mathbf{P})$ *with* $\|f\|_{L_p} \leqslant 1$ *there exists an interval* J *of* $\{1, \ldots, n\}$ *with* $J^c \neq \emptyset$ *and*

$$|J| \geqslant c(p, \varepsilon)n$$

such that for every $I \subseteq J$ *we have*

$$\mathbf{P}_I\big(\{\mathbf{x} \in \boldsymbol{\Omega}_I : |\mathbb{E}(f_{\mathbf{x}}) - \mathbb{E}(f)| \leqslant \varepsilon\}\big) \geqslant 1 - \varepsilon.$$

The Density Hales-Jewett Theorem, Probabilistic version

Theorem

Let k be positive integer and δ be a real with $0 < \delta \leqslant 1$. Then there exist an integer $n_0 = n_0(k, \delta)$ and a positive real $\theta = \theta(k, \delta)$ with the following property. For every $n \geqslant n_0$ and every family of measurable events $(A_t)_{t\in[k]^n}$ in a probability space (Ω, Σ, μ) such that $\mu(A_t) \geqslant \delta$ for all $t \in [k]^n$, there exists a variable word $w(v)$ of length n such that

$$\mu\Big(\bigcap_{a\in[k]} A_{w(a)}\Big) \geqslant \theta$$

Probabilistic versions

Theorem

Let k be positive integer and δ be a real with $0 < \delta \leqslant 1$. Then for each positive integer m there exist an integer $n_m = n(k, \delta, m)$ and a positive real $\theta_m = \theta(k, \delta, m)$ with the following property. For every $m \geqslant 1$ and every family of measurable events $(A_t)_{t \in [k]^{n_m}}$ in a probability space (Ω, Σ, μ) such that $\mu(A_t) \geqslant \delta$ for all $t \in [k]^{n_m}$, there exists an m-dimensional subspace S of $[k]^{n_m}$ such that

$$\mu\Big(\bigcap_{t \in S} A_t\Big) \geqslant \theta_m.$$

Probabilistic versions

Theorem (Dodos, Kanellopoulos and T.)

Let $k \geqslant 2$ and $0 < \delta \leqslant 1$. Then for every $m \geqslant 1$ there exist a positive integer $n_m = n(k, \delta, m)$ a positive real $\theta_m = \theta(k, \delta, m)$ having the following property. For every $m \geqslant 1$ and every family $\{A_t : t \in [k]^{n_m}\}$ of measurable events in a probability space (Ω, Σ, μ) such that $\mu(A_t) \geqslant \delta$ for all $t \in [k]^{n_m}$, there exists an m-dimensional subspace S of $[k]^{n_m}$ such that for every finite subset F of S we have that

$$\mu\Big(\bigcap_{t \in F} A_t\Big) \geqslant \theta_{|F|}.$$

- The proof of the above theorem requires a refinement of a partition result due to Furstenberg and Katznelson.

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