Monochromatic configurations in finite colorings of $\mathbb{N}$ - a dynamical approach

Joel P. Moreira
joel.moreira@northwestern.edu

Department of Mathematics
Northwestern University

November 22, 2017

Theorem (Schur)

For any finite coloring $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$ there exits $C \in \{C_1, \ldots, C_r\}$ and $x, y \in \mathbb{N}$ such that $\{x, y, x+y\} \subset C$.

Issai Schur

I. Schur, Jahresbericht der Deutschen Math, 1916

Theorem (Schur)

For any finite coloring $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$ *there exits* $C \in \{C_1, \ldots, C_r\}$ *and* $x, y \in \mathbb{N}$ *such that* $\{x, y, x+y\} \subset C$.

Issai Schur

Theorem (Schur, again)

For every r *there exists* $N \in \mathbb{N}$ *such that for every partition* $\{1, \ldots, N\} = C_1 \cup \cdots \cup C_r$ *there exist* $C \in \{C_1, \ldots, C_r\}$ *and* $x, y \in \{1, \ldots, N\}$ *such that* $\{x, y, x+y\} \subset C$.

I. Schur, Jahresbericht der Deutschen Math, 1916

Theorem (Schur)

For any finite coloring $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$ *there exits* $C \in \{C_1, \ldots, C_r\}$ *and* $x, y \in \mathbb{N}$ *such that* $\{x, y, x+y\} \subset C$.

Issai Schur

Theorem (Schur, again)

For every r *there exists* $N \in \mathbb{N}$ *such that for every partition* $\{1, \ldots, N\} = C_1 \cup \cdots \cup C_r$ *there exist* $C \in \{C_1, \ldots, C_r\}$ *and* $x, y \in \{1, \ldots, N\}$ *such that* $\{x, y, x+y\} \subset C$.

Schur used this to show that any large enough finite field contains nontrivial solutions to Fermat's equation $x^n + y^n = z^n$.

I. Schur, Jahresbericht der Deutschen Math, 1916

Theorem (van der Waerden)

For every $k \in \mathbb{N}$ and any finite partition $\mathbb{N} = C_1 \cup \cdots \cup C_r$, there exist $C \in \{C_1, \ldots, C_r\}$ and $x, y \in \mathbb{N}$ such that

$$\{x, x + y, x + 2y, \ldots, x + ky\} \subset C$$

B. van der Waerden, Nieuw. Arch. Wisk., 1927

A. Brauer, Sitz.ber. de Preus. Akad. Wiss., Phys.-Math. Kl., 1928

Theorem (van der Waerden)

For every $k \in \mathbb{N}$ and any finite partition $\mathbb{N} = C_1 \cup \cdots \cup C_r$, there exist $C \in \{C_1, \ldots, C_r\}$ and $x, y \in \mathbb{N}$ such that

$$\{x, x+y, x+2y, \ldots, x+ky\} \subset C$$

Theorem (Brauer)

For every $k \in \mathbb{N}$ and any finite partition $\mathbb{N} = C_1 \cup \cdots \cup C_r$, there exist $C \in \{C_1, \ldots, C_r\}$ and $x, y \in \mathbb{N}$ such that

$$\{x, y, x+y, x+2y, \ldots, x+ky\} \subset C$$

B. van der Waerden, Nieuw. Arch. Wisk., 1927

A. Brauer, Sitz.ber. de Preus. Akad. Wiss., Phys.-Math. Kl., 1928

Theorem (van der Waerden)

For every $k \in \mathbb{N}$ and any finite partition $\mathbb{N} = C_1 \cup \cdots \cup C_r$, there exist $C \in \{C_1, \ldots, C_r\}$ and $x, y \in \mathbb{N}$ such that

$$\{x, x+y, x+2y, \ldots, x+ky\} \subset C$$

Theorem (Brauer)

For every $k \in \mathbb{N}$ and any finite partition $\mathbb{N} = C_1 \cup \cdots \cup C_r$, there exist $C \in \{C_1, \ldots, C_r\}$ and $x, y \in \mathbb{N}$ such that

$$\{x, y, x+y, x+2y, \ldots, x+ky\} \subset C$$

B. van der Waerden, Nieuw. Arch. Wisk., 1927

A. Brauer, Sitz.ber. de Preus. Akad. Wiss., Phys.-Math. Kl., 1928

For every $k \in \mathbb{N}$ and any finite coloring $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$ there exist $C \in \{C_1, \ldots, C_r\}$ and $x, y \in \mathbb{N}$ such that

- **Schur:** $\{x, y, x+y\} \subset C.$
- **van der Waerden:** $\{x, x+y, x+2y, \ldots, x+ky\} \subset C.$
- **Brauer:** $\{x, y, x+y, x+2y, \ldots, x+ky\} \subset C.$

For every $k \in \mathbb{N}$ and any finite coloring $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$ there exist $C \in \{C_1, \ldots, C_r\}$ and $x, y \in \mathbb{N}$ such that

- **Schur:** $\{x, y, x + y\} \subset C.$
- **van der Waerden:** $\{x, x + y, x + 2y, \ldots, x + ky\} \subset C.$
- **Brauer:** $\{x, y, x + y, x + 2y, \ldots, x + ky\} \subset C.$

Problem

Let $f_1, \ldots, f_k : \mathbb{N}^m \to \mathbb{N}$. Under what conditions is it true that for any finite coloring $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$ there exist $C \in \{C_1, \ldots, C_r\}$ and $\mathbf{x} \in \mathbb{N}^m$ such that $\{f_1(\mathbf{x}), \ldots, f_k(\mathbf{x})\} \subset C$?

For every $k \in \mathbb{N}$ and any finite coloring $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$ there exist $C \in \{C_1, \ldots, C_r\}$ and $x, y \in \mathbb{N}$ such that

- **Schur:** $\{x, y, x+y\} \subset C.$
- **van der Waerden:** $\{x, x+y, x+2y, \ldots, x+ky\} \subset C.$
- **Brauer:** $\{x, y, x+y, x+2y, \ldots, x+ky\} \subset C.$

Problem

Let $f_1, \ldots, f_k : \mathbb{N}^m \to \mathbb{N}$. Under what conditions is it true that for any finite coloring $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$ there exist $C \in \{C_1, \ldots, C_r\}$ and $\mathbf{x} \in \mathbb{N}^m$ such that $\{f_1(\mathbf{x}), \ldots, f_k(\mathbf{x})\} \subset C$?

We say that $\{f_1(\mathbf{x}), \ldots, f_k(\mathbf{x})\}$ is a partition regular configuration.

Theorem (Folkman-Sanders-Rado)

For every $m \in \mathbb{N}$ and any finite coloring $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$ there exist $C \in \{C_1, \ldots, C_r\}$ and $x_0, \ldots, x_m \in \mathbb{N}$ such that

$$\begin{cases} x_0 & \\ x_1, & x_1 + x_0 \\ & \\ & \end{cases}$$

Theorem (Folkman-Sanders-Rado)

For every $m \in \mathbb{N}$ and any finite coloring $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$ there exist $C \in \{C_1, \ldots, C_r\}$ and $x_0, \ldots, x_m \in \mathbb{N}$ such that

$$\begin{cases} x_0 & & & \\ x_1, & x_1 + x_0 & & \\ x_2, & x_2 + x_1, & x_2 + x_0 & x_2 + x_1 + x_0 \\ & & & \\ & & & \end{cases}$$

Theorem (Folkman-Sanders-Rado)

For every $m \in \mathbb{N}$ and any finite coloring $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$ there exist $C \in \{C_1, \dots, C_r\}$ and $x_0, \dots, x_m \in \mathbb{N}$ such that

$$\left\{ \begin{array}{llll} x_0 & & & \\ x_1, & x_1 + x_0 & & \\ x_2, & x_2 + x_1, & x_2 + x_0 & x_2 + x_1 + x_0 \\ \vdots & \vdots & \vdots & \ddots \\ x_m, & x_m + x_{m-1}, & \dots & x_m + x_{m-1} + \cdots + x_0 \end{array} \right\} \subset C$$

Theorem (Deuber)

For every $m, k \in \mathbb{N}$ and any finite coloring $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$ there exist $C \in \{C_1, \ldots, C_r\}$ and $x_0, \ldots, x_m \in \mathbb{N}$ such that

$$\begin{cases} x_0, & \\ ix_0 + x_1, & i \in \{0, \ldots, k\} \\ & \\ & \end{cases}$$

W. Deuber, Math. Z, 1973

Theorem (Deuber)

For every $m, k \in \mathbb{N}$ and any finite coloring $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$ there exist $C \in \{C_1, \ldots, C_r\}$ and $x_0, \ldots, x_m \in \mathbb{N}$ such that

$$\begin{cases} x_0, & \\ ix_0 + x_1, & i \in \{0, \ldots, k\} \\ ix_0 + jx_1 + x_2, & i, j \in \{0 \ldots, k\} \\ & \\ & \end{cases}$$

W. Deuber, Math. Z, 1973

Theorem (Deuber)

For every $m, k \in \mathbb{N}$ and any finite coloring $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$ there exist $C \in \{C_1, \ldots, C_r\}$ and $x_0, \ldots, x_m \in \mathbb{N}$ such that

$$\left\{ \begin{array}{lr} x_0, & \\ ix_0 + x_1, & i \in \{0, \ldots, k\} \\ ix_0 + jx_1 + x_2, & i, j \in \{0 \ldots, k\} \\ \vdots & \vdots \\ i_0x_0 + \cdots + i_{m-1}x_{m-1} + x_m, & i_{m-1}, \ldots, i_0 \in \{0, \ldots, k\} \end{array} \right\} \subset C$$

W. Deuber, Math. Z, 1973

Theorem (Deuber)

For every $m, k, c \in \mathbb{N}$ and any finite coloring $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$ there exist $C \in \{C_1, \ldots, C_r\}$ and $x_0, \ldots, x_m \in \mathbb{N}$ such that

$$\left\{ \begin{array}{lr} cx_0, & \\ ix_0 + cx_1, & i \in \{0, \ldots, k\} \\ ix_0 + jx_1 + cx_2, & i, j \in \{0 \ldots, k\} \\ \vdots & \vdots \\ i_0x_0 + \cdots + i_{m-1}x_{m-1} + cx_m, & i_{m-1}, \ldots, i_0 \in \{0, \ldots, k\} \end{array} \right\} \subset C$$

In fact all partition regular linear configurations are contained in Deuber's theorem.

W. Deuber, Math. Z, 1973

Theorem (Deuber)

For every $m, k, c \in \mathbb{N}$ and any finite coloring $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$ there exist $C \in \{C_1, \ldots, C_r\}$ and $x_0, \ldots, x_m \in \mathbb{N}$ such that

$$\left\{ \begin{array}{lr} cx_0, & \\ ix_0 + cx_1, & i \in \{0, \ldots, k\} \\ ix_0 + jx_1 + cx_2, & i, j \in \{0 \ldots, k\} \\ \quad \vdots & \vdots \\ i_0x_0 + \cdots + i_{m-1}x_{m-1} + cx_m, & i_{m-1}, \ldots, i_0 \in \{0, \ldots, k\} \end{array} \right\} \subset C$$

In fact all partition regular linear configurations are contained in Deuber's theorem.

The complete classification of partition regular linear configurations was first obtained by R. Rado in 1933 using a different language.

W. Deuber, Math. Z, 1973

R. Rado, Math. Zeit., 1933

For every finite coloring $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$ there exist $C \in \{C_1, \ldots, C_r\}$ and $x, y \in \mathbb{N}$ such that:

Theorem (Furstenberg-Sárközy)

$\{x, x + y^2\} \subset C$.

H. Furstenberg, J. d'Analyse Math., 1977

A. Sárközy, Acta Math. Acad. Sci. Hungar., 1978

For every finite coloring $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$ there exist $C \in \{C_1, \ldots, C_r\}$ and $x, y \in \mathbb{N}$ such that:

Theorem (Furstenberg-Sárközy)

$\{x, x + f(y)\} \subset C$ for every $f \in \mathbb{Z}[x]$ with $f(0) = 0$.

H. Furstenberg, J. d'Analyse Math., 1977

A. Sárközy, Acta Math. Acad. Sci. Hungar., 1978

For every finite coloring $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$ there exist $C \in \{C_1, \ldots, C_r\}$ and $x, y \in \mathbb{N}$ such that:

Theorem (Furstenberg-Sárközy)

$\{x, x + f(y)\} \subset C$ *for every* $f \in \mathbb{Z}[x]$ *with* $f(0) = 0$.

Bergelson and Leibman extended this to

Theorem (Polynomial van der Waerden theorem)

Let $f_1, \ldots, f_k \in \mathbb{Z}[x]$ *be polynomials such that* $f_i(0) = 0$ *for all* $i = 1, \ldots, k$. *Then for any finite coloring of* $\mathbb{N} = C_1 \cup \cdots \cup C_r$ *there exist a color* $C \in \{C_1, \ldots, C_r\}$ *and* $x, y \in \mathbb{N}$ *such that*

$$\{x, x + f_1(y), x + f_2(y), \ldots, x + f_k(y)\} \subset C$$

H. Furstenberg, J. d'Analyse Math., 1977

A. Sárközy, Acta Math. Acad. Sci. Hungar., 1978

V. Bergelson and A. Leibman, , 1996

Theorem (Bergelson-Johnson-M.)

Let $m, c \in \mathbb{N}$ and, for each $i = 1, 2, \ldots, m$, let F_i be a finite set of polynomials $f : \mathbb{Z}^i \to \mathbb{Z}$ such that $f(\mathbf{0}) = 0$. Then for any finite coloring $\mathbb{N} = C_1 \cup \cdots \cup C_r$ there exists a color $C \in \{C_1, \ldots, C_r\}$ and $x_0, \ldots, x_m \in \mathbb{N}$ such that

$$\begin{cases} cx_0 & \\ f(x_0) + cx_1, & f \in F_1 \\ & \\ & \\ & \end{cases}$$

V. Bergelson, John Johnson and J. Moreira, J. Comb. Theory A, 2017

Theorem (Bergelson-Johnson-M.)

Let $m, c \in \mathbb{N}$ and, for each $i = 1, 2, \ldots, m$, let F_i be a finite set of polynomials $f : \mathbb{Z}^i \to \mathbb{Z}$ such that $f(\mathbf{0}) = 0$. Then for any finite coloring $\mathbb{N} = C_1 \cup \cdots \cup C_r$ there exists a color $C \in \{C_1, \ldots, C_r\}$ and $x_0, \ldots, x_m \in \mathbb{N}$ such that

$$\begin{cases} cx_0 & \\ f(x_0) + cx_1, & f \in F_1 \\ f(x_0, x_1) + cx_2, & f \in F_2 \\ \vdots & \vdots \\ & \end{cases}$$

V. Bergelson, John Johnson and J. Moreira, J. Comb. Theory A, 2017

Theorem (Bergelson-Johnson-M.)

Let $m, c \in \mathbb{N}$ and, for each $i = 1, 2, \ldots, m$, let F_i be a finite set of polynomials $f : \mathbb{Z}^i \to \mathbb{Z}$ such that $f(\mathbf{0}) = 0$. Then for any finite coloring $\mathbb{N} = C_1 \cup \cdots \cup C_r$ there exists a color $C \in \{C_1, \ldots, C_r\}$ and $x_0, \ldots, x_m \in \mathbb{N}$ such that

$$\left\{\begin{array}{lr} cx_0 & \\ f(x_0) + cx_1, & f \in F_1 \\ f(x_0, x_1) + cx_2, & f \in F_2 \\ \vdots & \vdots \\ f(x_0, \ldots, x_{m-1}) + cx_m, & f \in F_m \end{array}\right\} \subset C$$

V. Bergelson, John Johnson and J. Moreira, J. Comb. Theory A, 2017

Theorem (Bergelson-Johnson-M.)

Let $m, c \in \mathbb{N}$ and, for each $i = 1, 2, \ldots, m$, let F_i be a finite set of polynomials $f : \mathbb{Z}^i \to \mathbb{Z}$ such that $f(\mathbf{0}) = 0$. Then for any finite coloring $\mathbb{N} = C_1 \cup \cdots \cup C_r$ there exists a color $C \in \{C_1, \ldots, C_r\}$ and $x_0, \ldots, x_m \in \mathbb{N}$ such that

$$\left\{ \begin{array}{lr} cx_0 & \\ f(x_0) + cx_1, & f \in F_1 \\ f(x_0, x_1) + cx_2, & f \in F_2 \\ \vdots & \vdots \\ f(x_0, \ldots, x_{m-1}) + cx_m, & f \in F_m \end{array} \right\} \subset C$$

This is a joint extension of the polyomial van der Waerden theorem and Folkman's theorem. It contains Deuber's theorem as a special case.

V. Bergelson, John Johnson and J. Moreira, J. Comb. Theory A, 2017

Conjecture

For every finite coloring $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$ *there exist* $C \in \{C_1, \ldots, C_r\}$ *and* $x, y, z \in C$ *such that* $x^2 + y^2 = z^2$.

Conjecture

For every finite coloring $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$ there exist $C \in \{C_1, \ldots, C_r\}$ and $x, y, z \in C$ such that $x^2 + y^2 = z^2$.

- If one weakens the condition to $x, z \in C$ and $y \in \mathbb{N}$, the conjecture is still open. The analogue result in the ring of Gaussian integers was established by W. Sun.

W. Sun, https://arxiv.org/abs/1405.0241

Conjecture

For every finite coloring $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$ *there exist* $C \in \{C_1, \ldots, C_r\}$ *and* $x, y, z \in C$ *such that* $x^2 + y^2 = z^2$*.*

- If one weakens the condition to $x, z \in C$ and $y \in \mathbb{N}$, the conjecture is still open. The analogue result in the ring of Gaussian integers was established by W. Sun.
- The conjecture has been established when $r = 2$, but the proof relies (heavily) on the use of a computer.

W. Sun, https://arxiv.org/abs/1405.0241

M. Heule, O. Kullmann and V. Marek, https://arxiv.org/abs/1605.00723

Conjecture

For every finite coloring $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$ *there exist* $C \in \{C_1, \ldots, C_r\}$ *and* $x, y, z \in C$ *such that* $x^2 + y^2 = z^2$*.*

- If one weakens the condition to $x, z \in C$ and $y \in \mathbb{N}$, the conjecture is still open. The analogue result in the ring of Gaussian integers was established by W. Sun.
- The conjecture has been established when $r = 2$, but the proof relies (heavily) on the use of a computer.
- The conjecture is equivalent to ask if the configuration $\left\{2kmn, k(m^2 - n^2), k(m^2 + n^2)\right\}$ is partition regular.
- The conjecture is equivalent to ask if any multiplicatively syndetic set contains a Pythagorean triple.

W. Sun, https://arxiv.org/abs/1405.0241

M. Heule, O. Kullmann and V. Marek, https://arxiv.org/abs/1605.00723

For any finite partition $\mathbb{N} = C_1 \cup \cdots C_r$ there exist $C \in \{C_1, \ldots, C_r\}$ and...

- Schur: $x, y \in \mathbb{N}$ such that $\{x, y, x + y\} \subset C$.

For any finite partition $\mathbb{N} = C_1 \cup \cdots C_r$ there exist $C \in \{C_1, \ldots, C_r\}$ and...

- Schur: $x, y \in \mathbb{N}$ such that $\{x, y, x + y\} \subset C$.
- Corollary: $x, y \in \mathbb{N}$ such that $\{x, y, xy\} \subset C$.
  [Proof: restrict the coloring to $\{2^1, 2^2, 2^3, \ldots\}$.]

For any finite partition $\mathbb{N} = C_1 \cup \cdots C_r$ there exist $C \in \{C_1, \ldots, C_r\}$ and...

- Schur: $x, y \in \mathbb{N}$ such that $\{x, y, x+y\} \subset C$.
- Corollary: $x, y \in \mathbb{N}$ such that $\{x, y, xy\} \subset C$.
  [Proof: restrict the coloring to $\{2^1, 2^2, 2^3, \ldots\}$.]
- Hindman: $x, y, x', y' \in \mathbb{N}$ such that

$$\{x, y, x+y, \quad x', y', x'y'\} \subset C$$

  In other words, one can use the same color for both triples.

Hindman, Trans. Amer Math. Soc., 1979

For any finite partition $\mathbb{N} = C_1 \cup \cdots C_r$ there exist $C \in \{C_1, \ldots, C_r\}$ and...

- Schur: $x, y \in \mathbb{N}$ such that $\{x, y, x + y\} \subset C$.
- Corollary: $x, y \in \mathbb{N}$ such that $\{x, y, xy\} \subset C$.
  [Proof: restrict the coloring to $\{2^1, 2^2, 2^3, \ldots\}$.]
- Hindman: $x, y, x', y' \in \mathbb{N}$ such that

  $$\{x, y, x + y, \quad x', y', x'y'\} \subset C$$

  In other words, one can use the same color for both triples.
- A variation on Hindman's method gives $x = x'$.

Hindman, Trans. Amer Math. Soc., 1979

Conjecture

For every finite coloring $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$ *there exist* $C \in \{C_1, \ldots, C_r\}$ *and* $x, y \in \mathbb{N}$ *such that* $\{x, y, x+y, xy\} \subset C$.

An analogue in finite fields was recently established by B. Green and T. Sanders.

B. Green, T. Sanders, Disc. Anal., 2016

Conjecture

For every finite coloring $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$ *there exist* $C \in \{C_1, \ldots, C_r\}$ *and* $x, y \in \mathbb{N}$ *such that* $\{x, y, x+y, xy\} \subset C$.

An analogue in finite fields was recently established by B. Green and T. Sanders.

Conjecture

For every finite coloring $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$ *there exist* $C \in \{C_1, \ldots, C_r\}$ *and* $x, y \in \mathbb{N}$ *such that* $\{\quad x+y, xy\} \subset C$.

B. Green, T. Sanders, Disc. Anal., 2016

Conjecture

For every finite coloring $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$ *there exist* $C \in \{C_1, \ldots, C_r\}$ *and* $x, y \in \mathbb{N}$ *such that* $\{x, y, x+y, xy\} \subset C$.

An analogue in finite fields was recently established by B. Green and T. Sanders.

Theorem (M.)

For every finite coloring $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$ *there exist* $C \in \{C_1, \ldots, C_r\}$ *and* $x, y \in \mathbb{N}$ *such that* $\{x, x+y, xy\} \subset C$.

B. Green, T. Sanders, Disc. Anal., 2016

J. Moreira, Ann. of Math., 2017

Theorem (M.)

Let $s \in \mathbb{N}$ and, for each $i = 1, \ldots, s$, let $F_i \subset \mathbb{Z}[x_1, \ldots, x_i]$ be a finite set of polynomials such that with 0 constant term. Then for any finite coloring $\mathbb{N} = C_1 \cup \cdots \cup C_r$ there exist $C \in \{C_1, \ldots, C_r\}$ and $x_0, \ldots, x_s \in \mathbb{N}$ such that for every $i, j \in \mathbb{Z}$ with $0 \leq j < i \leq s$ and every $f \in F_{i-j}$ we have

$$x_0 \cdots x_j + f(x_{j+1}, \ldots, x_i) \in C$$

J. Moreira, Ann. of Math., 2017

Corollary

For every finite coloring $\mathbb{N} = C_1 \cup C_2 \cup \cdots \cup C_r$ *there exist* $C \in \{C_1, \ldots, C_r\}$ *and* $x, y, z, t, w \in \mathbb{N}$ *such that*

$$\left\{ \begin{array}{ccccc} x & & & & \\ xy, & x+y & & & \\ xyz, & x+yz, & xy+z & & \\ xyzt, & x+yzt, & xy+zt, & xyz+t & \\ xyztw, & x+yztw, & xy+ztw, & xyz+tw & xyzt+w \end{array} \right\} \subset C$$

Corollary

Let $k \in \mathbb{N}$ *and* $c_1, \ldots, c_k \in \mathbb{Z} \setminus \{0\}$ *be such that* $c_1 + \cdots + c_k = 0$. *Then for any finite coloring of* $\mathbb{N}$ *there exist pairwise distinct* $a_0, \ldots, a_k \in \mathbb{N}$, *all of the same color, such that*

$$c_1 a_1^2 + \cdots + c_k a_k^2 = a_0.$$

In particular, there exist $x, y, z \in C$ such that $x^2 - y^2 = z$.

- Given $E \subset \mathbb{N}$, its upper density is

$$\overline{d}(E) := \limsup_{N\to\infty} \frac{\left|E \cap \{1,\ldots,N\}\right|}{N}$$

- Upper density is shift invariant: $\bar{d}(E-n) = \bar{d}(E)$ for all n.
- $\bar{d}(A \cup B) \leq \bar{d}(A) + \bar{d}(B)$.
- In particular, for any finite coloring $\mathbb{N} = C_1 \cup \cdots \cup C_r$ some C_i has positive upper density.

A *measure preserving system* is a triple (X, μ, T), where

- (X, μ) is a probability space.
- $T : X \to X$ preserves μ, i.e., for any (measurable) set $A \subset X$,

$$\mu\big(T^{-1}A\big) = \mu\big(\{x \in X : Tx \in A\}\big) = \mu(A).$$

A *measure preserving system* is a triple (X, μ, T), where

- (X, μ) is a probability space.
- $T : X \to X$ preserves μ, i.e., for any (measurable) set $A \subset X$,

$$\mu\big(T^{-1}A\big) = \mu\big(\{x \in X : Tx \in A\}\big) = \mu(A).$$

Example

- Let $X = [0, 1]$, $\mu =$ Lebesgue measure, $T : x \mapsto x + \alpha \bmod 1$, for some $\alpha \in \mathbb{R}$.
- Let $X = [0, 1]$, $\mu =$ Lebesgue measure, $T : x \mapsto 2x \bmod 1$.

A *measure preserving system* is a triple (X, μ, T), where

- (X, μ) is a probability space.
- $T : X \to X$ preserves μ, i.e., for any (measurable) set $A \subset X$,

$$\mu\big(T^{-1}A\big) = \mu\big(\{x \in X : Tx \in A\}\big) = \mu(A).$$

Example

- Let $X = [0,1]$, $\mu =$ Lebesgue measure, $T : x \mapsto x + \alpha \bmod 1$, for some $\alpha \in \mathbb{R}$.
- Let $X = [0,1]$, $\mu =$ Lebesgue measure, $T : x \mapsto 2x \bmod 1$.

Not quite an example: $X = \mathbb{N}$, $\mu = \bar{d}$ and $T : x \mapsto x + 1$.

A *measure preserving system* is a triple (X, μ, T), where

- (X, μ) is a probability space.
- $T : X \to X$ preserves μ, i.e., for any (measurable) set $A \subset X$,

$$\mu\big(T^{-1}A\big) = \mu\big(\{x \in X : Tx \in A\}\big) = \mu(A).$$

Theorem (Furstenberg Correspondence Principle)

Let $E \subset \mathbb{N}$. There exists a measure preserving system (X, μ, T) and a set $A \subset X$ such that $\mu(A) = \overline{d}(E)$ and

$$\overline{d}\big((E-n_1)\cap(E-n_2)\cap\cdots\cap(E-n_k)\big) \geq \mu\big(T^{-n_1}A\cap T^{-n_2}A\cap\cdots\cap T^{-n_k}A\big)$$

for any $n_1, \ldots, n_k \in \mathbb{N}$.

Szemerédi's theorem follows from the correspondence principle together with:

Theorem (Furstenberg's multiple recurrence theorem)

Let (X, μ, T) be a measure preserving system and let $A \subset X$ with $\mu(A) > 0$. Then for every k

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu\big(A \cap T^{-n}A \cap T^{-2n}A \cap \cdots \cap T^{-kn}A\big) > 0$$

Szemerédi's theorem follows from the correspondence principle together with:

Theorem (Furstenberg's multiple recurrence theorem)

Let (X, μ, T) *be a measure preserving system and let* $A \subset X$ *with* $\mu(A) > 0$*. Then for every* k

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu\big(A \cap T^{-n}A \cap T^{-2n}A \cap \cdots \cap T^{-kn}A\big) > 0$$

Theorem (von Neumann's Ergodic Theorem)

Let (X, μ, T) *be a measure preserving system and let* $A \subset X$*. Then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu\big(A \cap T^{-n}A\big) \geq \mu(A)^2$$

Let $E \subset \mathbb{N}$ with $\bar{d}(E) > 0$.

- $\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T^{-n^2} A) \geq \mu(A)^2$
  - Corresponds to $\{x, x + y^2\} \subset E$.

Let $E \subset \mathbb{N}$ with $\bar{d}(E) > 0$.

- $\displaystyle \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T^{-n^2} A) \geq \mu(A)^2$
  - Corresponds to $\{x, x+y^2\} \subset E$.
- $\displaystyle \lim_{N\to\infty} \frac{1}{\pi(N)} \sum_{p \leq N} \mu(A \cap T^{-(p-1)} A \cap \cdots \cap T^{-k(p-1)} A) > 0$
  - Corresponds to $\{x, x+(p-1), \cdots, x+k(p-1)\} \subset E$.

Let $E \subset \mathbb{N}$ with $\bar{d}(E) > 0$.

- $\displaystyle \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mu(A \cap T^{-n^2}A) \geq \mu(A)^2$
  - Corresponds to $\{x, x + y^2\} \subset E$.

- $\displaystyle \lim_{N\to\infty} \frac{1}{\pi(N)} \sum_{p \leq N} \mu(A \cap T^{-(p-1)}A \cap \cdots \cap T^{-k(p-1)}A) > 0$
  - Corresponds to $\{x, x + (p-1), \cdots, x + k(p-1)\} \subset E$.

- For any $f \in \mathbb{Z}[x]$ with $f(1) = 0$,
  $\displaystyle \lim_{N\to\infty} \frac{1}{\pi(N)} \sum_{p \leq N} \mu(A \cap T^{-f(p)}A \cap \cdots \cap T^{-kf(p)}A) > 0$
  - Corresponds to $\{x, x + f(p), \cdots, x + kf(p)\} \subset E$.

Want to show that for any finite partition $\mathbb{N} = C_1 \cup \cdots \cup C_r$, some C_i contains a configuration $\{n + m, nm\}$.

Want to show that for any finite partition $\mathbb{N} = C_1 \cup \cdots \cup C_r$, some C_i contains a configuration $\{n + m, nm\}$.

$$\{n + m, nm\} \subset C \iff m \in (C - n) \cap (C/n),$$

where

$$C - n = \{m \in \mathbb{N} : m + n \in C\} \qquad C/n = \{m \in \mathbb{N} : mn \in C\}$$

Want to show that for any finite partition $\mathbb{N} = C_1 \cup \cdots \cup C_r$, some C_i contains a configuration $\{n+m, nm\}$.

$$\{n+m, nm\} \subset C \iff m \in (C-n) \cap (C/n),$$

where

$$C - n = \{m \in \mathbb{N} : m+n \in C\} \qquad C/n = \{m \in \mathbb{N} : mn \in C\}$$

Thus we need a notion of density invariant under addition and multiplication.

$$d(C-n) = d(C) \qquad \text{and} \qquad d(C/n) = d(C)$$

Want to show that for any finite partition $\mathbb{N} = C_1 \cup \cdots \cup C_r$, some C_i contains a configuration $\{n + m, nm\}$.

$$\{n + m, nm\} \subset C \iff m \in (C - n) \cap (C/n),$$

where

$$C - n = \{m \in \mathbb{N} : m + n \in C\} \qquad C/n = \{m \in \mathbb{N} : mn \in C\}$$

Thus we need a notion of density invariant under addition and multiplication.

$$d(C - n) = d(C) \qquad \text{and} \qquad d(C/n) = d(C)$$

Unfortunately, no such density exists on $\mathbb{N}$.

The semigroup generated by addition and multiplication – the semigroup of all affine transformations $x \mapsto ax + b$ with $a, b \in \mathbb{N}$ – is not *amenable*.

- Denote by $\mathcal{A}_{\mathbb{Q}}$ the group of all affine transformations of $\mathbb{Q}$:

$$\mathcal{A}_{\mathbb{Q}} := \left\{ x \mapsto ax + b \;:\; a, b \in \mathbb{Q},\; a \neq 0 \right\}$$

- This is the semidirect product of the groups $(\mathbb{Q}, +)$ and $(\mathbb{Q}^*, \times)$; hence it is solvable, and in particular amenable.

- Denote by $\mathcal{A}_{\mathbb{Q}}$ the group of all affine transformations of $\mathbb{Q}$:

$$\mathcal{A}_{\mathbb{Q}} := \big\{ x \mapsto ax + b \ : \ a, b \in \mathbb{Q}, \ a \neq 0 \big\}$$

- This is the semidirect product of the groups $(\mathbb{Q}, +)$ and $(\mathbb{Q}^*, \times)$; hence it is solvable, and in particular amenable.

Proposition

There exists an upper density $\bar{d} : \mathcal{P}(\mathbb{Q}) \to [0,1]$ which is invariant under both addition and multiplication, i.e.,

$$\bar{d}(E) = \bar{d}(E - x) = \bar{d}(E/x).$$

Equivalently, there exists a sequence $(F_N)_{N \in \mathbb{N}}$ of finite subsets of $\mathbb{Q}$ such that for every $x \in \mathbb{Q} \setminus \{0\}$,

$$\lim_{N\to\infty} \frac{|F_N \cap (F_N + x)|}{|F_N|} = \lim_{N\to\infty} \frac{|F_N \cap (F_N x)|}{|F_N|} = 1$$

Vitaly Bergelson

Theorem (Bergelson, M.)

If $C \subset \mathbb{Q}$ has $\bar{d}(C) > 0$, then there exist

- *"many" $x, y \in \mathbb{Q}$ such that $\{x + y, xy\} \subset C$;*

V. Bergelson, M., Erg. Theo. Dyn. Syst., 2016

V. Bergelson, M., Erg. Theo. Dyn. Syst., 2018

Vitaly Bergelson

Theorem (Bergelson, M.)

If $C \subset \mathbb{Q}$ has $\bar{d}(C) > 0$, then there exist

- *"many" $x, y \in \mathbb{Q}$ such that $\{x + y, xy\} \subset C$;*
- *"many" $x \in \mathbb{Q}$, $y \in \mathbb{N}$ such that $\{x + y, xy\} \subset C$.*

V. Bergelson, M., Erg. Theo. Dyn. Syst., 2016

V. Bergelson, M., Erg. Theo. Dyn. Syst., 2018

The proofs have three ingredients:

- The existence of a doubly invariant upper density $\bar{d}$,
- A modified Furstenberg Correspondence principle,
- A “mixed” ergodic theorem:

The proofs have three ingredients:

- The existence of a doubly invariant upper density $\bar{d}$,
- A modified Furstenberg Correspondence principle,
- A "mixed" ergodic theorem:

Theorem (Bergelson, M.)

Let $E \subset \mathbb{Q}$ and assume that $\bar{d}(E) > 0$. Then

$$\lim_{N\to\infty} \frac{1}{|F_N|} \sum_{x\in F_N} \bar{d}\Big((E-x) \ \cap \ (E/x)\Big) > 0$$

For $u \in \mathbb{Q}$, let $M_u : x \mapsto ux$ and $A_u : x \mapsto u + x$.

Theorem (Bergelson, M.)

Let $(U_g)_{g \in \mathcal{A}_{\mathbb{Q}}}$ be a unitary representation of $\mathcal{A}_{\mathbb{Q}}$ on a Hilbert space H with no fixed vectors. Then for every $f \in H$,

$$\lim_{N \to \infty} \frac{1}{|F_N|} \sum_{u \in F_N} M_u A_u f = 0$$

For $u \in \mathbb{Q}$, let $M_u : x \mapsto ux$ and $A_u : x \mapsto u + x$.

Theorem (Bergelson, M.)

Let $(U_g)_{g \in \mathcal{A}_\mathbb{Q}}$ be a unitary representation of $\mathcal{A}_\mathbb{Q}$ on a Hilbert space H with no fixed vectors. Then for every $f \in H$,

$$\lim_{N \to \infty} \frac{1}{|F_N|} \sum_{u \in F_N} M_u A_u f = 0$$

The key is to realize the map $g : \mathbb{Q} \to \mathcal{A}_\mathbb{Q}$ taking u to $M_u A_u$ as a "polynomial".

For $u \in \mathbb{Q}$, let $M_u : x \mapsto ux$ and $A_u : x \mapsto u + x$.

Theorem (Bergelson, M.)

Let $(U_g)_{g \in \mathcal{A}_\mathbb{Q}}$ be a unitary representation of $\mathcal{A}_\mathbb{Q}$ on a Hilbert space H with no fixed vectors. Then for every $f \in H$,

$$\lim_{N \to \infty} \frac{1}{|F_N|} \sum_{u \in F_N} M_u A_u f = 0$$

The key is to realize the map $g : \mathbb{Q} \to \mathcal{A}_\mathbb{Q}$ taking u to $M_u A_u$ as a "polynomial".

- Let $\Delta_h^A g(u) = g(u)^{-1} g(u + h)$ and $\Delta_h^M g(u) = g(u)^{-1} g(uh)$.

For $u \in \mathbb{Q}$, let $M_u : x \mapsto ux$ and $A_u : x \mapsto u + x$.

Theorem (Bergelson, M.)

Let $(U_g)_{g \in \mathcal{A}_{\mathbb{Q}}}$ be a unitary representation of $\mathcal{A}_{\mathbb{Q}}$ on a Hilbert space H with no fixed vectors. Then for every $f \in H$,

$$\lim_{N \to \infty} \frac{1}{|F_N|} \sum_{u \in F_N} M_u A_u f = 0$$

The key is to realize the map $g : \mathbb{Q} \to \mathcal{A}_{\mathbb{Q}}$ taking u to $M_u A_u$ as a "polynomial".

- Let $\Delta_h^A g(u) = g(u)^{-1} g(u + h)$ and $\Delta_h^M g(u) = g(u)^{-1} g(uh)$.
- We have that for all $h, \tilde{h} \in \mathbb{Q}$,

$$\Delta_h^A \Delta_{\tilde{h}}^M g \text{ is constant!}$$

Let $\mathcal{A}_{\mathbb{N}}^{-} := \{x \mapsto ax + b : a \in \mathbb{N}, b \in \mathbb{Z}\}$.

Theorem (A topological correspondence principle)

There exists an $\mathcal{A}_{\mathbb{N}}^{-}$-topological system $(X, (T_g)_{g \in \mathcal{A}_{\mathbb{N}}^{-}})$ with a dense set of additively minimal points, such that each map $T_g : X \to X$ is open and injective, and with the property that for any finite coloring $\mathbb{N} = C_1 \cup \cdots \cup C_r$ there exists an open cover $X = U_1 \cup \cdots \cup U_r$ such that for any $g_1, \ldots, g_k \in \mathcal{A}_{\mathbb{N}}^{-}$ and $t \in \{1, \ldots, r\}$,

$$\bigcap_{\ell=1}^{k} T_{g_\ell}(U_t) \neq \emptyset \qquad \Longrightarrow \qquad \mathbb{N} \cap \bigcap_{\ell=1}^{k} g_\ell(C_t) \neq \emptyset$$

Let $\mathcal{A}_{\mathbb{N}}^{-} := \{x \mapsto ax + b : a \in \mathbb{N}, b \in \mathbb{Z}\}$.

Theorem (A topological correspondence principle)

There exists an $\mathcal{A}_{\mathbb{N}}^{-}$-topological system $(X, (T_g)_{g \in \mathcal{A}_{\mathbb{N}}^{-}})$ with a dense set of additively minimal points, such that each map $T_g : X \to X$ is open and injective, and with the property that for any finite coloring $\mathbb{N} = C_1 \cup \cdots \cup C_r$ there exists an open cover $X = U_1 \cup \cdots \cup U_r$ such that for any $g_1, \ldots, g_k \in \mathcal{A}_{\mathbb{N}}^{-}$ and $t \in \{1, \ldots, r\}$,

$$\bigcap_{\ell=1}^{k} T_{g_\ell}(U_t) \neq \emptyset \qquad \Longrightarrow \qquad \mathbb{N} \cap \bigcap_{\ell=1}^{k} g_\ell(C_t) \neq \emptyset$$

- In particular, if $A_y^{-1} U_t \cap M_y^{-1} U_t \neq \emptyset$, then $C_t \supset \{x + y, xy\}$ for some x, where $A_y : x \mapsto x + y$ and $M_y : x \mapsto xy$.

Theorem

For every "nice" topological system $(X, (T_g)_{g \in \mathcal{A}_{\mathbb{N}}^-})$ *and every open cover* $X = U_1 \cup U_2 \cup \cdots \cup U_r$ *there exist* $U \in \{U_1, \ldots, U_r\}$ *and* $y \in \mathbb{N}$ *such that*

$$A_y^{-1} U \cap M_y^{-1} U \neq \emptyset$$

Theorem

For every "nice" topological system $(X, (T_g)_{g \in \mathcal{A}_{\mathbb{N}}^-})$ *and every open cover* $X = U_1 \cup U_2 \cup \cdots \cup U_r$ *there exist* $U \in \{U_1, \ldots, U_r\}$ *and* $y \in \mathbb{N}$ *such that*

$$A_y^{-1} U \cap M_y^{-1} U \neq \emptyset$$

Equivalently

$$M_y A_{-y} U \cap U \neq \emptyset.$$

Theorem

For every "nice" topological system $(X, (T_g)_{g \in \mathcal{A}_{\mathbb{N}}^{-}})$ *and every open cover* $X = U_1 \cup U_2 \cup \cdots \cup U_r$ *there exist* $U \in \{U_1, \ldots, U_r\}$ *and* $y \in \mathbb{N}$ *such that*

$$A_y^{-1} U \cap M_y^{-1} U \neq \emptyset$$

Equivalently

$$M_y A_{-y} U \cap U \neq \emptyset.$$

Idea

Find a sequence $B_1, B_2, \ldots$ of non-empty sets such that

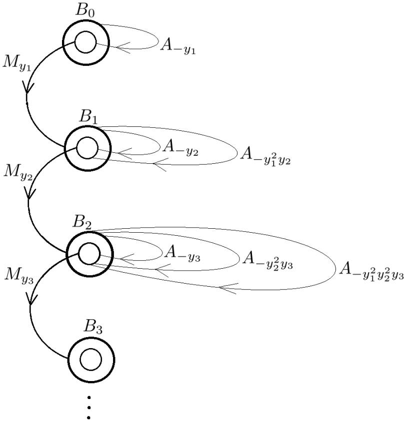
- Each B_i is contained in a single color U_j;
- For every $i < j$ there is $y \in \mathbb{N}$ such that $B_j \subset M_y A_{-y}(B_i)$

We want to find a sequence $B_1, B_2, \ldots$ of non-empty sets such that

- Each B_i is contained in a single color;
- For every $i < j$ there is $y \in \mathbb{N}$ such that $B_j \subset M_y A_{-y}(B_i)$

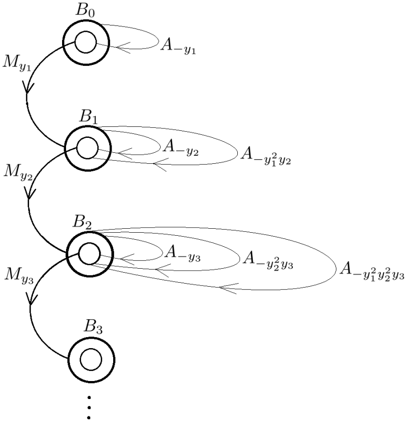
We want to find a sequence $B_1, B_2, \dots$ of non-empty sets such that

- Each B_i is contained in a single color;
- For every $i < j$ there is $y \in \mathbb{N}$ such that $B_j \subset M_y A_{-y}(B_i)$



We want to find a sequence $B_1, B_2, \dots$ of non-empty sets such that

- Each B_i is contained in a single color;
- For every $i < j$ there is $y \in \mathbb{N}$ such that $B_j \subset M_y A_{-y}(B_i)$



To run the iterative construction we use the following version of van der Waerden's theorem:

Theorem

Let $(X, (T_g)_{g \in \mathcal{A}_{\mathbb{N}}^-})$ be a "nice" topological system and $B \subset X$ open and non-empty. Then for every $k \in \mathbb{N}$ there exists $y \in \mathbb{N}$ such that

$$B \cap A_{-y} B \cap A_{-2y} B \cap \cdots \cap A_{-ky} B \neq \emptyset$$

Questions?