The approximate Ramsey property of classes of finite dimensional normed spaces

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Outline

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  3. the class $\{\ell_p^n\}_{n\in\mathbb{N}}$;
  4. the class of polyhedral spaces.
- Our proof of the (ARP) of $\{\ell_\infty^n\}_{n\in\mathbb{N}}$ uses the Dual Ramsey Theorem by Graham ad Rothschild, while the (ARP) of $\{\ell_p^n\}_{n\in\mathbb{N}}$, $p\neq 2,\infty$ can be proved by the version of the Dual Ramsey Theorem for equipartitions (open) or its approximate version (true, with a non-combinatorial proof).

- In a precise way, the (ARP) of the classes $\{\ell_p^n\}_{n\in\mathbb{N}}$ can be seen as the factorization theorem for Grassmannians over $\mathbb{R},\mathbb{C}$ that corresponds to the Graham-Leeb-Rothschild Theorem on Grassmannians over a finite field;

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- the (ARP) with a multidimensional version of the Borsuk-Ulam Theorem.

This is a joint work with D Bartosova, M. Lupini and B. Mbombo, and V. Ferenczi, B. Mbombo and S. Todorcevic.

Some basics

- A normed space is a vector space X (over $\mathbb{F} = \mathbb{R}, \mathbb{C}$) together with a norm $\|\cdot\| : X \to [0, \infty[$; when $\|\cdot\|$ is complete, X is a Banach space;

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- $S_X = \{x\in X \,:\, \|x\|=1\}$, $B_X := \{x\in X \,:\, \|x\|\le 1\}$ are the unit ball of X and the unit sphere of X, respectively;
- Given X,Y let $\binom{Y}{X}$ be the collection of all subspaces of Y isometric to X. When X is finite dimensional, its unit ball is compact; We endow $\binom{Y}{X}$ with with the Hausdorff metric metric on it:

$$d(X_0,X_1)=\max\{\max_{x_0\in B_{X_0}}\min_{x_1\in B_{X_1}}\|x_0-x_1\|_Y, \max_{x_1\in B_{X_1}}\min_{x_0\in B_{X_0}}\|x_1-x_0\|_Y\}$$

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Definition

A collection $\mathscr{F}$ of finite dimensional normed spaces has the weak Approximate Ramsey Property (ARP) when for every $F, G \in \mathscr{F}$ and $\varepsilon > 0$ there exists $H \in \mathscr{F}$ such that

$$H \longrightarrow (G)^F_\varepsilon,$$

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that is, every continuous coloring c of $\binom{H}{F}$ ε-stabilizes in $\binom{\widehat{G}}{F}$ for some $\widehat{G} \in \binom{H}{G}$, i.e.,

$$\operatorname{osc}(c \restriction \binom{\widehat{G}}{F}) = \sup_{F_0, F_1 \in \binom{\widehat{G}}{F}} |c(F_0) - c(F_1)| < \varepsilon.$$

Some basics

- given two Banach spaces X and Y, an isometric embedding is a linear map $T : X \to Y$ such that $\|Tx\|_Y = \|x\|_X$; let $\mathrm{Emb}(X, Y)$ be the space of all isometric embeddings from X into Y endowed with the operator distance:

$$d(T, U) = \|T - U\| := \sup_{\|x\|_X \leq 1} \|T(x) - U(x)\|_Y.$$

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This is a particular instance of a more general definition for metric structures.

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6. *The class of all finite dimensional normed spaces (B-LA-L-Mb).*

Previous Known results

1. Odell-Rosenthal-Schlumprecht proved that for every $1 \leq p \leq \infty$, every $m \in \mathbb{N}$ and every $\varepsilon > 0$ there is $n \in \mathbb{N}$ such that

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3. Matoušek-Rödl proved the first result for $1 \leq p < \infty$ combinatorially (using spreads).

Consequences in topological dynamics

Definition

A Banach space X is called approximately ultrahomogeneous (aUH) when for every finite dimensional subspace F of X, every $\varepsilon > 0$ and every isometric embedding $\gamma : F \to X$ there is some global isometry I of X such that $\|I \upharpoonright F - \gamma\| < \varepsilon$.

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- *The group of isometries* $\mathrm{Iso}(X)$ *with its strong operator topology is extremely amenable;*
- *The class* $\mathrm{Age}(X)$ *of finite dimensional subspaces of X has the (ARP).*

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It is not difficult to see that if X is compact, then every open covering is ε-fat for some $\varepsilon > 0$.

Theorem (ARP for $\{\ell_p^n\}_n$)

For every $1 \le p \le \infty$*, every integers* d, m *and* r *and every* $\varepsilon > 0$ *there is some* $\mathbf{n}_p(d,m,r,\varepsilon)$ *such that for every* ε*-fat open covering* $\mathscr{U}$ *of* $\mathrm{Emb}(\ell_p^d, \ell_p^n)$ *with cardinality at most* r *there exists* $\varrho \in \mathrm{Emb}(\ell_p^m, \ell_p^n)$ *such that*

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Problem

Does there exists $\mathbf{n}_p(d,m,r,\varepsilon)$ independent of ε?

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3. By the KPT correspondence, we have the (ARP) of $\mathrm{Age}(\ell_2)$, and (trivially) of $\{\ell_2^n\}_{n\in\mathbb{N}}$.

Hints of the proofs. $p \neq 2, \infty$:

1. For $1 \leq p \neq 2, \infty$, the group $\mathrm{Iso}(L_p[0,1])$ is, by Banach-Lamperti, topologically isomorphic to the semidirect product of $L^0([0,1], \{-1,1\})$ and the non-singular transformations $\mathrm{Aut}^*([0,1])$; both of them are Lévy (Giordano-Pestov), so $\mathrm{Iso}(L_p[0,1])$ is extremely amenable; notice that all the groups are topologically isomorphic;

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2. Lusky proved that $L_p[0,1]$, $p \neq 4, 6, 8, \dots, \infty$ is (aUH); this gives the (ARP) of $\mathrm{Age}(L_p[0,1])$ for those p's.

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4. When $p = 4, 6, 8, \dots$ there are arbitrarily large finite dimensional subspaces X of L_p well complemented in L_p having isometric copies badly complemented. The coloring asking if a copy of $\widehat{X} \in \binom{\ell_p^n}{X}$ is well or badly complemented is a bad (discrete) coloring.

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The proof by B-LA-L-Mb goes as follows:

1. First of all, one establishes the (ARP) of $\{\ell_\infty^n\}_n$ as a consequence of the Dual Ramsey Theorem by Graham and Rothschild;

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1. First of all, one establishes the (ARP) of $\{\ell_\infty^n\}_n$ as a consequence of the Dual Ramsey Theorem by Graham and Rothschild;
2. Then one proves the (ARP) of the class of finite dimensional polyhedral spaces Pol; recall that a f.d. polyhedral space is a space whose unit ball is a polytope, i.e. it has finitely many extreme points; this is done by using the injective envelope of P: this is a pair $(\gamma, \ell_\infty^{n_P})$ such that $\gamma_P : P \to \ell_\infty^{n_P}$ is an isometric embedding with the property that any other $\gamma : P \to \ell_\infty^n$ factors through γ_P; this allows to reduce colorings of $\mathrm{Emb}(P, \ell_\infty^n)$ to colorings of $\mathrm{Emb}(\ell_\infty^{n_P}, \ell_\infty^n)$.

Hints of the proofs. $p = \infty$

3. An arbitrary f.d. space is limit of polyhedral spaces;
4. For every $\delta, \varepsilon > 0$ and every f.d. X, Y there is some f.d. Z and an isometric embedding $I : Y \to Z$ such that

$$I \circ \mathrm{Emb}_\delta(X,Y) \subseteq (\mathrm{Emb}(X,Z))_{\varepsilon+\delta}.$$

The Dual Ramsey Theorem

Definition

Let $(S, <_S)$ *and* $(T, <_T)$ *be two linearly ordered sets. A surjection* $\theta : S \to T$ *is called a rigid-surjection when* $\min \theta^{-1}(t_0) < \min \theta^{-1}(t_1)$ *for every* $t_0 < t_1$ *in* T*. Let* $\mathrm{Epi}(S,T)$ *be collection of all those surjections.*

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Theorem (Dual Ramsey Theorem; Graham and Rothschild)

For every finite linearly ordered sets S and T, and $r \in \mathbb{N}$ there exists $n \geq \#T$ such that every r-coloring of $\mathrm{Epi}(n,S)$ *has a monochromatic set of the form* $\mathrm{Epi}(T,S)\circ\sigma$ *for some* $\sigma \in \mathrm{Epi}(n,T)$.

Definition

Let $\mathcal{E}_{n\times k}$ be the collection of all $n\times k$ matrices representing (in the unit bases of $\mathbb{F}^k$ ad $\mathbb{F}^n$) a linear isometry between ℓ_∞^k and ℓ_∞^n.

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Given $\varepsilon>0$, let $\mathscr{N}$ be a finite ε-dense subset of the unit ball $B_{\ell_1^k}$

1. containing 0 and the unit vectors u_i, and
2. such that for every non-zero $v\in B_{\ell_1^k}$ there is $w\in\mathscr{N}$ such that $\|v-w\|_1<\varepsilon$ and $\|w\|_1<\|v\|_1$. e.g., for large $l\varepsilon\geq 1$,

$$\mathscr{N}=(\{\pm\frac{i}{kl}\}_{i\leq kl})^k\cap B_{\ell_1^k}$$

Let $<$ be any total ordering on $\mathscr{N}$ such that $v < w$ when $\|v\|_1 < \|w\|_1$. We order n canonically.

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Let $\Phi : \mathrm{Epi}(n, \mathscr{N}) \to \mathscr{E}_{n\times k}$ be defined for $\sigma : \{1, \ldots, n\} \to \mathscr{N}$ as the $n \times k$-matrix A_σ whose ξ-row vector, $1 \le \xi \le n$, is $\sigma(\xi)$.

It is easy to see that $\Phi(\sigma) \in \mathscr{E}_{n\times k}$.

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It is easy to see that $\Phi(\sigma) \in \mathscr{E}_{n\times k}$. To simplify, suppose that $\mathbb{F} = \mathbb{R}$.

Proposition

There is a finite set $\Gamma \subseteq \mathscr{E}_{n\times k}$ such that for every other $A \in \mathscr{E}_{n\times k}$ there exists $B \in \Gamma$ such that

$$A^{\mathrm{t}} B = \mathrm{Id}_k.$$

We order now $\Delta := \mathscr{N} \times \Gamma$ lexicographically, where Δ is arbitrarily ordered. Given now k, m, a number of colors r, we use apply the DR theorem to $\mathscr{N}$ and Δ to find the corresponding n.

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$$c \circ \Phi : \mathrm{Epi}(n, \mathscr{N}) \to r$$

Let $\varrho \in \mathrm{Epi}(n, \Delta)$ such that c is constant on $\mathrm{Epi}(\Delta, \mathscr{N}) \circ \varrho$. Let now $R \in \mathscr{E}_{n\times m}$ be the matrix whose ξ-column is Av where $\varrho(\xi) = (v, A)$.

Proposition

For every $B \in \mathscr{E}_{m\times d}$ there exists $\sigma \in \mathrm{Epi}(\Delta, \mathscr{N})$ such that $\|RB - \Phi(\varrho \circ \sigma)\|_\infty < \varepsilon$.

We want to follow the same strategy than for $p = \infty$. Let $\mathscr{E}^p_{n \times d}$ be the collection of matrices that represent isometric embeddings from ℓ_p^d into ℓ_p^n; they are characterized by the fact that each column vector is p-normalized and on each row there is at most one non-zero entry.

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Definition

A mapping $T : n \to \Delta$ is called an ε-equipartition, $\varepsilon \geq 0$ when

$$\frac{n}{\#\Delta}(1-\varepsilon) \leq \#F^{-1}(\delta) \leq \frac{n}{\#\Delta}(1+\varepsilon)$$

for every $\delta \in \Delta$. Let $\mathrm{Equi}_\varepsilon(n, \Delta)$ be the set of all ε-equipartions, and $\mathrm{Equi}(n, \Delta)$ be the rigid-surjections (0-)equipartitions.

Equipartitions

Theorem (Approximate Ramsey property for ε-equipartitions)

Let $d, m, r \in \mathbb{N}$, $\varepsilon_0 \geq 0$ and $\varepsilon_1, \delta > 0$, and let $\varepsilon_2 \geq 0$ be such that $(1-\varepsilon_2) \leq (1-\varepsilon_0)(1-\varepsilon_1) \leq (1+\varepsilon_0)(1+\varepsilon_1) \leq (1+\varepsilon_2)$. Then there is n such that for every coloring $c : \mathrm{Equi}_{\varepsilon_2}(n,d) \to \{1,\dots,r\}$ there exists $R \in \mathrm{Equi}_{\varepsilon_1}(n,m)$ and $1 \leq i \leq r$ such that

$$\mathrm{Equi}_{\varepsilon_0}(m,d) \circ R \subseteq (c^{-1}(i))_{2\varepsilon_0(1+\varepsilon_1)+\delta(1+\varepsilon_0)}.$$

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Problem (Dual Ramsey for equipartitions)

Suppose that $d|m$, and r is arbitrary. Does there exist $m|n$ such that every r-coloring of $\mathrm{Equi}(n,d)$ has a monochromatic set of the form $\mathrm{Equi}(m,d) \circ \sigma$ for some $\sigma \in \mathrm{Equi}(n,m)$?

Concentration

We prove the previous result by using concentration of measure of the *Hamming cube* Δ^n.

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$$\alpha_X(\varepsilon) := 1 - \inf\{\mu(A_\varepsilon) : \mu(A) \geq \frac{1}{2}\}.$$

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and normal Lévy when there are $c_1, c_2 > 0$ *such that*

$$\alpha_{X_n}(\varepsilon) \leq c_1 e^{-c_2 \varepsilon^2 n}.$$

It is known that

$$\alpha_{(\Delta^n,d,\mu)}(\varepsilon) \leq e^{-\frac{1}{8}\varepsilon^2 n},$$

where d is the normalized Hamming distance

$$d(f,g) := \frac{1}{n}\#(f \neq g)$$

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Proposition

$(\mathrm{Equi}_\varepsilon(n,\Delta), d, \mu)_n$ *is asymptotically normal Lévy.*

Rephrasing the Dual Ramsey Theorem

We say that $0-1$-valued $n \times k$-matrix is boolean if the column vectors are non null and the supports of column vectors of A form a partition of the target set n.

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We denote them by $\mathscr{I}^{\mathrm{ba}}_{n\times k}$. We call a boolean matrix A ordered when the support of the i^{th} column of A starts before the support of the $(i+1)^{\mathrm{th}}$ column of A. Given a boolean $n \times k$-matrix, let $\mathrm{corr}(A) \in \mathrm{S}_k$ be the unique permutation matrix (i.e. automorphism of the Boolean algebra $\mathscr{P}(k)$) such that $A \cdot \mathrm{corr}(A)$ is ordered.

0-1 valued matrices

Proposition (DRT, embedding version)

For every k, m and r there is $n \geq k$ such that every r-coloring $c : \mathscr{I}^{\mathrm{ba}}_{n\times k} \to r$ factors

$$\begin{array}{ccc} R \cdot \mathscr{I}^{\mathrm{ba}}_{m\times k} & \xrightarrow{\;c\;} & r \\ & \searrow^{\mathrm{corr}} \; \circlearrowleft & \uparrow \widehat{c} \\ & & \mathrm{S}_k \end{array}$$

for some ordered boolean $n \times m$-matrix R. Observe that $\mathrm{corr}(R \cdot A) = \mathrm{corr}(A)$.

Matrices over a finite field $\mathbb{F}$

Let $\mathscr{I}_{n\times k}(\mathbb{F})$ be the collection of $n \times k$-matrices of rank k.

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Given $A \in \mathscr{I}_{n\times k}(\mathbb{F})$, let red$(A) \in \mathrm{GL}(\mathbb{F}^k)$ be such that $A \cdot \mathrm{red}(A)$ is in Reduced Column Echelon Form (RCEF).

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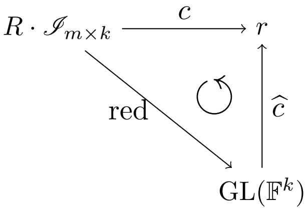
$$\begin{pmatrix} 0 & 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & & & \vdots \\ 0 & 0 & 0 & \cdots & \cdots & 0 \\ 1 & 0 & 0 & \cdots & \cdots & 0 \\ * & 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & & & \vdots \\ * & 0 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & \cdots & 0 \\ * & * & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & & & \vdots \\ * & * & 0 & 0 & \cdots & 0 \\ * & * & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \end{pmatrix}$$

Theorem

For every $k, m \in \mathbb{N}$ *and* $r \in \mathbb{N}$ *there exists* $n \geq k$ *such that every coloring* $c : \mathscr{I}_{n\times k}(\mathbb{F}) \to r$ *factors*



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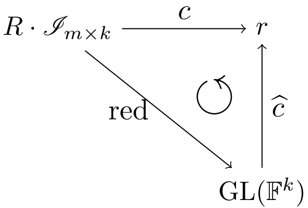

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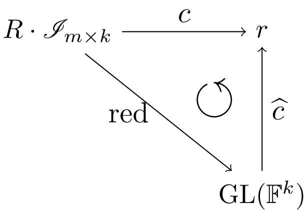

for some $R \in \mathscr{I}_{n\times m}(\mathbb{F})$ *in RCEF.*

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Theorem (Graham-Leeb-Rothschild)

Suppose that $\mathbb{F}$ *is a finite field. For every* $k, m \in \mathbb{N}$ *and* $r \in \mathbb{N}$ *there exists* $n \geq k$ *such that every* r*-coloring of* $\mathrm{Gr}(k, \mathbb{F}^n)$*, the* k*-Grassmannians of* $\mathbb{F}^n$*, has a monochromatic set of the form* $\mathrm{Gr}(k, V)$ *for some* $V \in \mathrm{Gr}(m, \mathbb{F}^n)$*.*

What for full rank matrices with entries in $\mathbb{R}, \mathbb{C}$?

There is a natural factorization result, but now approximative. The set of matrices is endowed with natural metrics, to each full-rank matrix A we associate a norm $\tau(A)$. It is proved that this mapping, with the right metrics is 1-Lipschitz. We obtain a factorization theorem for full rank matrices.

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- Let $\mathscr{N}_k$ be the polish space of all norms on $\mathbb{F}^k$, and let $\mathscr{N}_k^p$ be the set of all norms $N \in \mathscr{N}_k$ such that $(\mathbb{F}^k, N)$ can be isometrically embedded into E_p. $\mathrm{GL}(\mathbb{F}^k)$ acts on $\mathscr{N}_k$, $A \cdot N(x) := N(A^{-1}x)$;

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- The k-Banach-Mazur compactum $\mathscr{B}_k$ is the orbit space $(\mathscr{N}_k,\omega)/\!/\mathrm{GL}(\mathbb{F}^k)$. Let $\mathscr{B}_k^p := (\mathscr{N}_k^p,\omega)/\!/\mathrm{GL}(\mathbb{F}^k)$;

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- We define the p-gap (opening) metric $\Lambda_p(V,W)$ between $V,W \in \mathrm{Gr}(k,\mathbb{F}^n)$ as the Hausdorff distance (with respect to M) between the unit balls of $(V,\|\cdot\|_p)$ and $(W,\|\cdot\|_p)$.

- the p-Kadets metric is the compatible Gromov-Hausdorff distance on $\mathscr{B}_k^p$ defined by

$$\gamma_p(\mathbf{M}, \mathbf{N}) := \inf_{T,U} \Lambda_p(TX, UY)$$

where the infimum runs over all isometric embeddings $T : (\mathbb{F}^k, M) \to E_p$, $U : (\mathbb{F}^k, N) \to E_p$.

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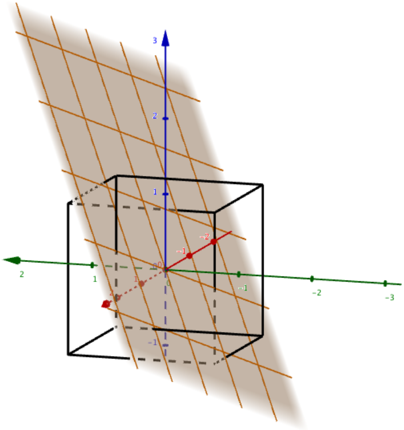
- Let $\tau_p : (\mathrm{Gr}(k, \mathbb{F}^n), \Lambda_p) \to (\mathscr{B}_k^p, \gamma_p)$ be the 1-Lipschitz map that assigns to $V \in \mathrm{Gr}(k, \mathbb{F}^n)$ the "isometric type" of $(V, \|\cdot\|_p)$.

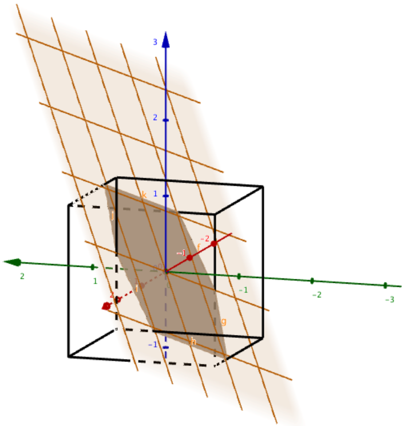
The mapping τ_{BM} for the sup norm

Given a Plane (so $k = 2$)



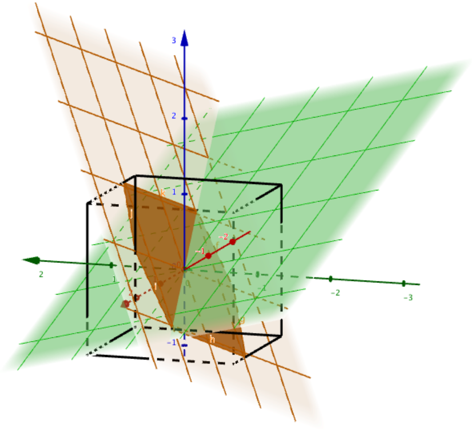
The mapping τ_{BM} for the sup norm

we consider its section with the cube



The mapping τ_{BM} for the sup norm

Another section with the same shape



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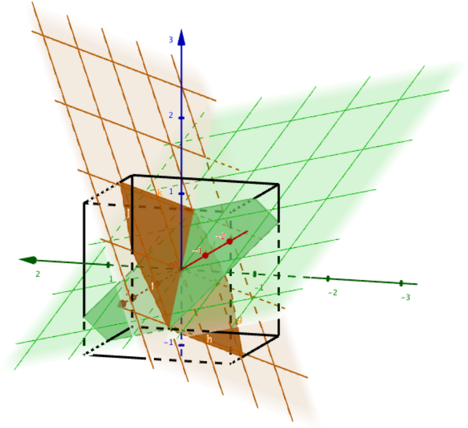
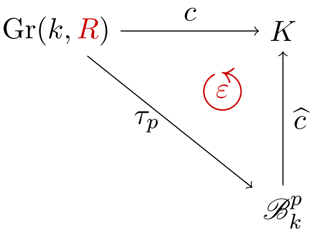


GLR for $\mathbb{F} = \mathbb{R}, \mathbb{C}$

Theorem (GLR Theorem for $\mathbb{R}$, p-version)

Let $p \neq 4, 6, 8, \ldots$. For every k, m $\varepsilon > 0$ and every (K, d_K) compact metric there is n such that for every 1-Lipschitz coloring $c : (\mathrm{Gr}(k, \mathbb{F}^n), \Lambda_p) \to (K, d_K)$ there is some $R \in \mathrm{Gr}(m, \mathbb{F}^n)$ such that $(R, \|\cdot\|_p)$ is isometric to ℓ_p^m, and a 1-Lipschitz $\widehat{c} : (\mathscr{B}_k^p, \gamma_p) \to (K, d_K)$ such that



GLR Theorem for $\mathbb{R}, \mathbb{C}$, Euclidean version

Theorem

For every k, m, $C > 0$, $\varepsilon > 0$ and every (K, d_K) compact metric there is $n \geq k$ such that for every norm M on $\mathbb{R}^n$, every C-Lipschitz coloring of $(\mathrm{Gr}(k, \mathbb{R}^n), \Lambda_M)$ by (K, d_K) ε-stabilizes in some $\mathrm{Gr}(k, V)$, that is, there exists $V \in \mathrm{Gr}(m, \mathbb{F}^n)$ such that

$$\mathrm{diam}_K(c(\mathrm{Gr}(k, V))) < \varepsilon$$

GLR Theorem for $\mathbb{R}, \mathbb{C}$, Euclidean version

Theorem

For every k, m, $C > 0$, $\varepsilon > 0$ and every (K, d_K) compact metric there is $n \geq k$ such that for every norm M on $\mathbb{R}^n$, every C-Lipschitz coloring of $(\mathrm{Gr}(k, \mathbb{R}^n), \Lambda_M)$ by (K, d_K) ε-stabilizes in some $\mathrm{Gr}(k, V)$, that is, there exists $V \in \mathrm{Gr}(m, \mathbb{F}^n)$ such that

$$\mathrm{diam}_K(c(\mathrm{Gr}(k, V))) < \varepsilon$$

This is consequence of Dvoretzky's Theorem and the GLR Theorem for $p = 2$.

Thank You!