Ramsey theory, trees, and ultrafilters

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Part II

The second involves developing Ramsey theory on trees in order to find bounds on the Ramsey degrees of homogeneous structures. A name for an ultrafilter is used in a key step.

Erdős-Rado Canonical Equivalence Relations

Given $k \geq 1$ and $I \subseteq k := \{0, \ldots, k-1\}$, the canonical equivalence relation E_I is defined as follows:

For $\bar{a} = \{a_0, \ldots, a_{k-1}\}$ and $\bar{b} = \{b_0, \ldots, b_{k-1}\}$,

$$\bar{a} \; E_I \; \bar{b} \Leftrightarrow \forall i \in I(a_i = b_i).$$

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Thm. (Erdős-Rado) Given $k \geq 1$ and E an equivalence relation on $[\mathbb{N}]^k$, there is an infinite $M \subseteq \mathbb{N}$ and an $I \subseteq k$ such that $E \restriction M = E_I \restriction M$.

This is really a theorem about infinitely many colors.

Part I: Tukey theory of ultrafilters

$\mathcal{U}$ and $\mathcal{V}$ denote ultrafilters on countable base sets.

Def. $\mathcal{V}$ is Tukey reducible to $\mathcal{U}$ ($\mathcal{V} \le_T \mathcal{U}$) if there is a map $f : \mathcal{U} \to \mathcal{V}$ such that each f-image of a filter base for $\mathcal{U}$ is a filter base for $\mathcal{V}$.

$$\mathcal{U} \equiv_T \mathcal{V} \text{ iff } \mathcal{U} \le_T \mathcal{V} \text{ and } \mathcal{V} \le_T \mathcal{U}.$$

The Tukey equivalence class of an ultrafilter $\mathcal{U}$, denoted $[\mathcal{U}]_T$, is called its Tukey type. These are exactly the cofinal types of ultrafilters.

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3. Tukey types of ultrafilters are a coarsening of the Rudin-Keisler types.

   $\mathcal{V} \leq_{RK} \mathcal{U}$ iff $\exists f : \omega \to \omega$ such that $\{f(U) : U \in \mathcal{U}\}$ generates $\mathcal{V}$.

$$\mathcal{V} \leq_{RK} \mathcal{U} \Longrightarrow \mathcal{V} \leq_T \mathcal{U}.$$

   Thus, every Tukey type is partitioned into isomorphism (RK) classes.

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   Thus, every Tukey type is partitioned into isomorphism (RK) classes.
4. Some very interesting Ramsey theory has developed from this study.

Main types of results

1. Differences between $\leq_T$ and $\leq_{RK}$.
2. When a Tukey non-top ultrafilter exists.
3. Canonical cofinal maps.
4. When $\leq_T$ implies $\leq_{RK}$ or even $\leq_{RB}$.
5. Structures embedded into the Tukey types of ultrafilters.
6. Exact structures in the Tukey (and Rudin-Keisler) types of ultrafilters.

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This talk will focus on 6, with a nod to 3.

Ramsey ultrafilters are Tukey minimal

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Blass previously showed that Ramsey ultrafilters are RK minimal.

Initial structures and connections with Ramsey theory

Def. A collection of Tukey types of nonprincipal ultrafilters $(C, \leq_T)$ is an initial Tukey structure if for each $[\mathcal{U}]_T \in C$, for each $\mathcal{V} \leq_T \mathcal{U}$, also $[\mathcal{V}]_T \in C$.

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Similarly, one can investigate initial Rudin-Keisler structures.

Initial Tukey and RK types of weakly Ramsey ultrafilters

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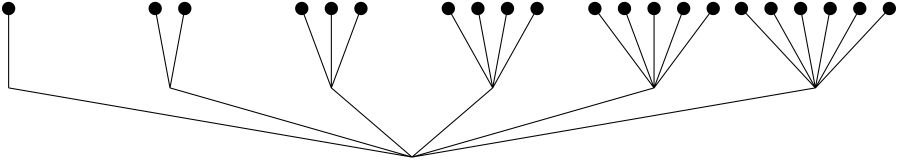
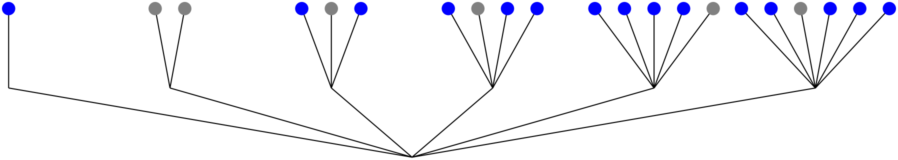
Members of the topological Ramsey space $\mathcal{R}_1$





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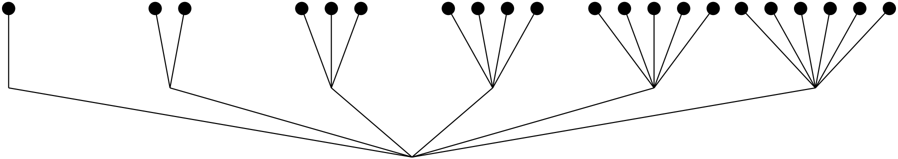
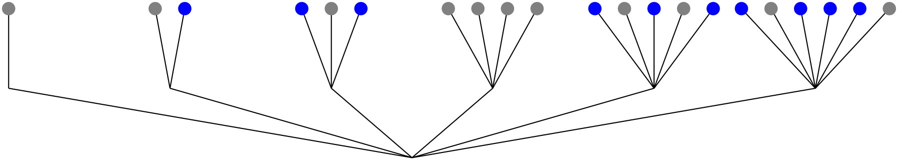
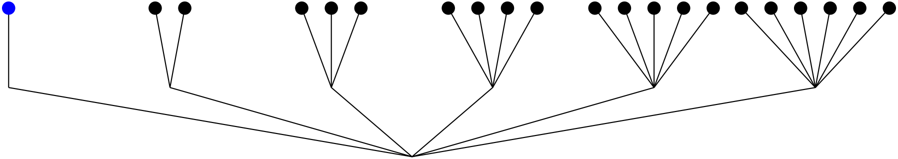




Analogue of $[\mathbb{N}]^1$: First Approximations



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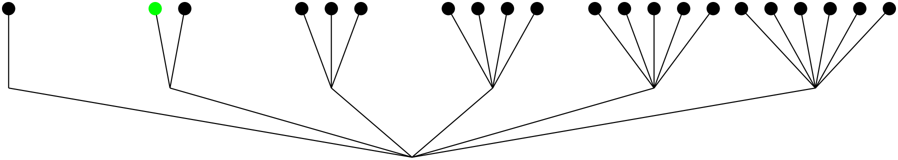

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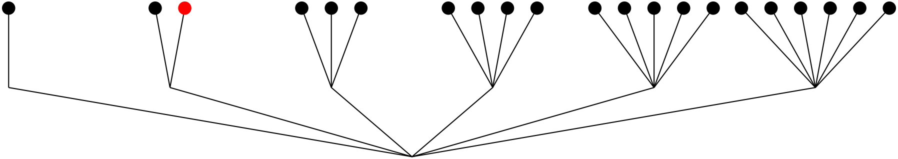

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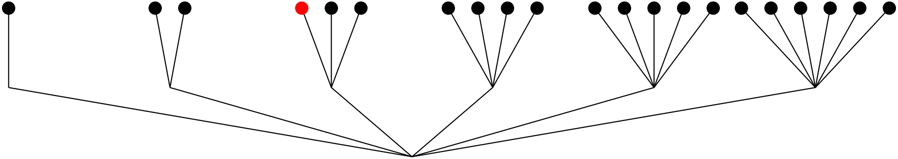

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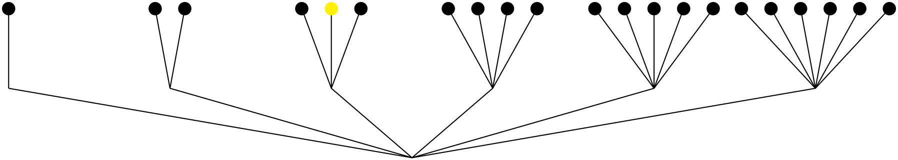

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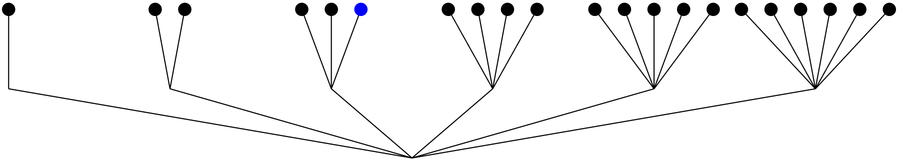
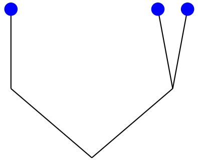


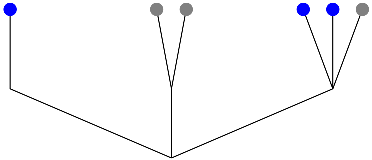
Erdős-Rado Analogue. Given any equivalence relation E on first approximations to members of the space $\mathcal{R}_1$, there is a member $X \in \mathcal{R}_1$ such that the restriction of E to X is canonical: given by projections via one of the three subtrees of $\{(), (0), (00)\}$.

Analogue of $[\mathbb{N}]^2$: Second Approximations

The set of second approximations to members of $\mathcal{R}_1$ are subtrees of the following form.



The following are a couple of examples.



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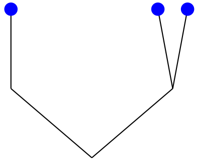

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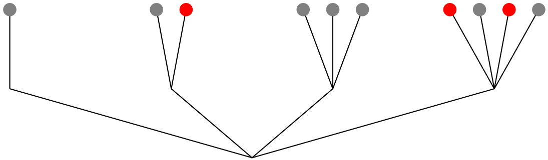
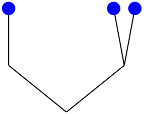


Analogue of Erdős-Rado for Second Approximations

Thm. Given an equivalence relation E on the set of second approximations to members of the space $\mathcal{R}_1$, there is a subtree T of



and there is some member $X \in \mathcal{R}_1$ such that E is canonized by the projection map π_T.

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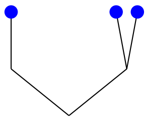

and there is some member $X \in \mathcal{R}_1$ such that E is canonized by the projection map π_T.

So there are $(2^1 + 1)(2^2 + 1)$ canonical equivalence relations on the second approximations.

More generally, we obtained in [DT2] a theorem which canonizes equivalence relations on any barrier in $\mathcal{R}_1$ via projection maps to subtrees.

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Other canonical equivalence relations on trees were proved in order to obtain the following structural results.

Initial chains in Tukey and Rudin-Keisler structures

Thm. [D/T 2,3] For each $1 \leq \alpha < \omega_1$, there is an ultrafilter $\mathcal{U}_\alpha$ such that the Tukey structure below $\mathcal{U}_\alpha$ forms the linear order $(\alpha + 1)^*$.

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This is the same for the initial Rudin-Keisler structure below $\mathcal{U}_\alpha$.

Other partial orders as initial structures

Thm. [D/Mijares/Trujillo] $([\omega]^{<\omega}, \subseteq)$ is an initial Tukey structure, and moreover consists of Tukey types of p-points.

Let $k \geq 1$ and let $\mathcal{K}_i$, $i < k$, be any Fraïssé classes of finite relational structures with the Ramsey property (and OPFAP). Then the set of isomorphism classes of $(\mathcal{K}_0, \dots, \mathcal{K}_{k-1})$, partially ordered by embedding, is realized as the initial Rudin-Keisler structure of some p-point.

This work encompasses as special cases the k-arrow, not $(k+1)$-arrow ultrafilters of Baumgartner and Taylor, and the n-square ultrafilters of Blass.

Initial Tukey and RK structures of size $\mathfrak{c}$

Thm. [D 1-3] There are ultrafilters with interesting partition properties, not p-points, which yield an initial Rudin-Keisler structure which is a linear order of size $\mathfrak{c}$ which is isomorphic to a non-standard model of $\mathbb{N}$.

The initial Tukey structure contains a copy of the initial Rudin-Keisler structure but also contains more.

General outline for the preceding results

1. Construct a topological Ramsey space which forces the desired ultrafilter. In most cases, these were constructed to be dense subsets of previously known forcings.

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Remark: In [DT2,3] and [DMT], the structure of the isomorphism classes inside the Tukey types are completely classified.

Part II

Ramsey Theory on Trees and applications to big Ramsey degrees of homogeneous structures.

Finite Structural Ramsey Theory

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Some Fraïssé classes of finite structures with the Ramsey property:
Boolean algebras, vector spaces over a finite field, ordered graphs, ordered hypergraphs, ordered graphs omitting k-cliques, ordered metric spaces, and many others.

Small Ramsey Degrees

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The classes of finite graphs, hypergraphs, graphs omitting k-cliques, etc., have small Ramsey degrees.

Ramsey Theory on Infinite Structures

Def. (Kechris, Pestov, Todorcevic 2005)
Let $\mathcal{K}$ be a Fraïssé class and $\mathbf{F} = \mathbf{Flim}(\mathcal{K})$. $\mathbf{F}$ has finite big Ramsey degrees if for each $A \in \mathcal{K}$, there is a finite number $T(A, \mathcal{K})$ such that for any coloring of $\binom{\mathbf{F}}{\mathbf{A}}$ into finitely many colors, there is a substructure $\mathbf{F}'$ of $\mathbf{F}$, with $\mathbf{F}' \cong \mathbf{F}$, in which $\binom{\mathbf{F}'}{\mathbf{A}}$ take no more than $T(A, \mathcal{K})$ colors.

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Infinite structures known to have finite big Ramsey degrees: The rationals (Devlin 1979); the Rado graph (Sauer 2006); the countable ultrametric Urysohn space (Nguyen Van Thé 2008); the $\mathbb{Q}_n$ and $\mathbf{S(2)}$, $\mathbf{S(3)}$ (Laflamme, NVT, Sauer 2010), and a few others,

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Connections with Topological Dynamics

Thm. (Kechris/Pestov/Todorcevic 2005) Aut(Flim $\mathcal{K}$) has the *fixed point on compacta property* if and only if $\mathcal{K}$ has the Ramsey property and consists of rigid elements.

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(Zucker 2017) Characterized universal completion flows of Aut(Flim $\mathcal{K}$) whenever Flim $\mathcal{K}$ admits a big Ramsey structure (big Ramsey degrees).

Strong Trees and Milliken's Theorem

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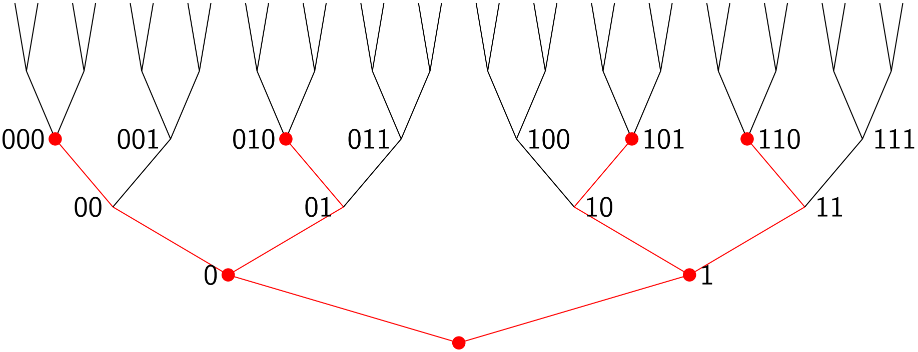
Strong Trees and Milliken's Theorem

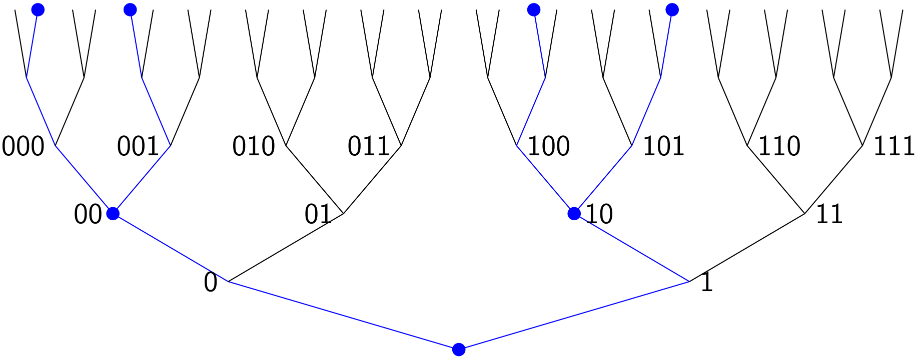
A Ramsey theorem on strong trees due to Milliken plays a central role in Devlin's and Sauer's results. A new colored version was key in [L/NVT/S 2010].

A tree $T \subseteq 2^{<\omega}$ is a strong tree iff it is either isomorphic to $2^{<\omega}$ or to $2^{\leq k}$ for some finite k.

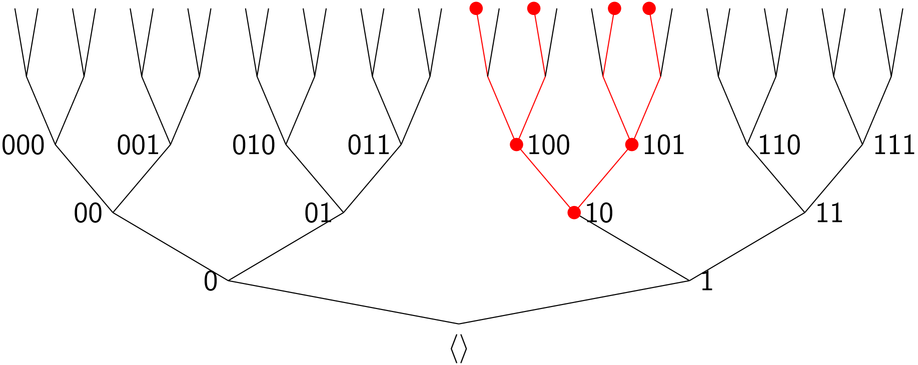
Strong Subtree $\cong 2^{\leq 2}$, Ex. 1



Strong Subtree $\cong 2^{\leq 2}$, Ex. 2



Strong Subtree $\cong 2^{\leq 2}$, Ex. 3



A Ramsey Theorem for Strong Trees

Thm. (Milliken 1979) Let $k \geq 0$, $l \geq 2$, and a coloring of all the subtrees of $2^{<\omega}$ which are isomorphic to $2^{\leq k}$ into l colors. Then there is an infinite strong subtree $S \subseteq 2^{<\omega}$ such that all copies of $2^{\leq k}$ in S have the same color.

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Milliken's Theorem builds on the Halpern-Läuchli Theorem.

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III Finish.

Part I: Strong Coding Trees

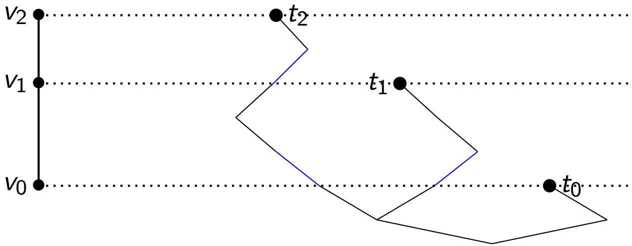
Nodes in Trees can Code Graphs

Let A be a graph. Enumerate the vertices of A as $\langle v_n : n < N \rangle$.

A set of nodes $\{t_n : n < N\}$ in $2^{<\omega}$ codes A if and only if for each pair $m < n < N$,

$$v_n \ E \ v_m \Leftrightarrow t_n(|t_m|) = 1.$$

The number $t_n(|t_m|)$ is called the passing number of t_n at t_m.



First Approach: Strong Triangle-Free Trees

Finite strong triangle-free trees are finite trees in with a unary predicate for distinguished nodes coding vertices, and which branch as much as possible, subject to no branch being extendable to a node coding a triangle.

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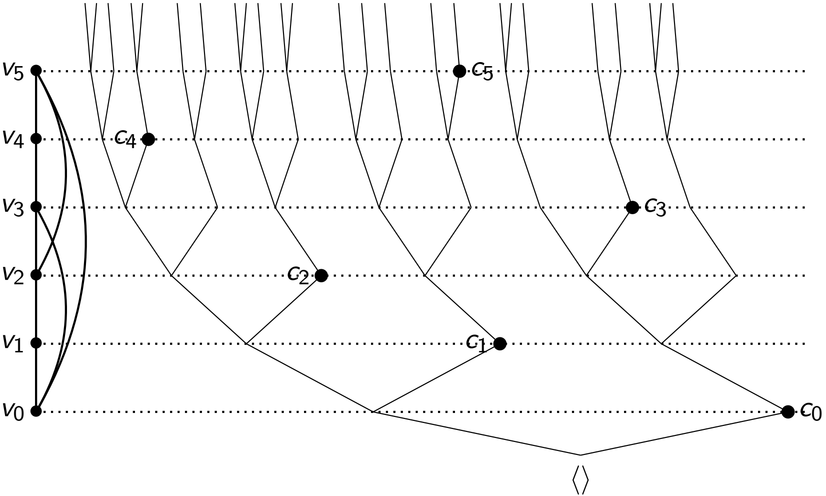
Infinite strong triangle-free trees have coding nodes which are dense and which code the universal triangle-free graph.

Strong triangle-free tree $\mathbb{S}$



Almost sufficient

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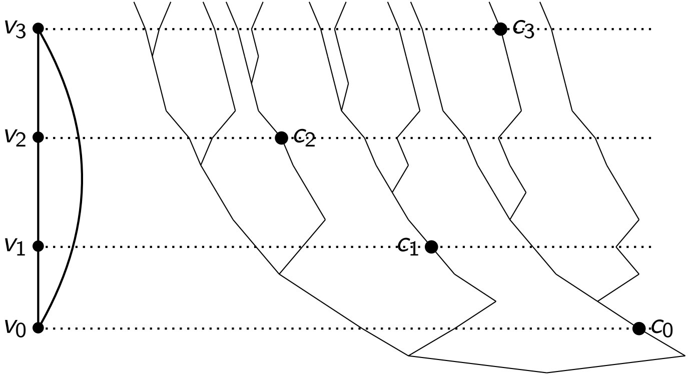
To get around this, we stretch and skew the trees so that at most one coding or one splitting node occurs at each level.

These skewed trees densely coding $\mathcal{H}_3$ are called strong coding trees.

Strong coding tree $\mathbb{T}$



Write $T \leq \mathbb{T}$ if T is a subtree of $\mathbb{T}$ strongly similar to T.
Every tree $T \leq \mathbb{T}$ is a strong coding tree: Its coding nodes are dense and code $\mathcal{H}_3$, and the "zip up" forms a strong triangle-free tree.

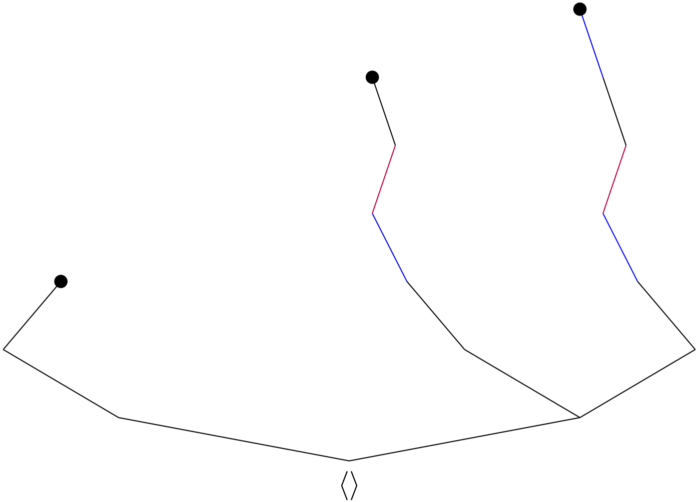
Ramsey Theorem for Strictly Similar Antichains

Theorem. (D.) Let A be a finite antichain of coding nodes. Associate A with the tree it induces, and let c color all strictly similar copies of A in T into finitely many colors.

Then there is a strong coding tree $S \leq T$ in which all strictly similar copies of A in S have the same color.

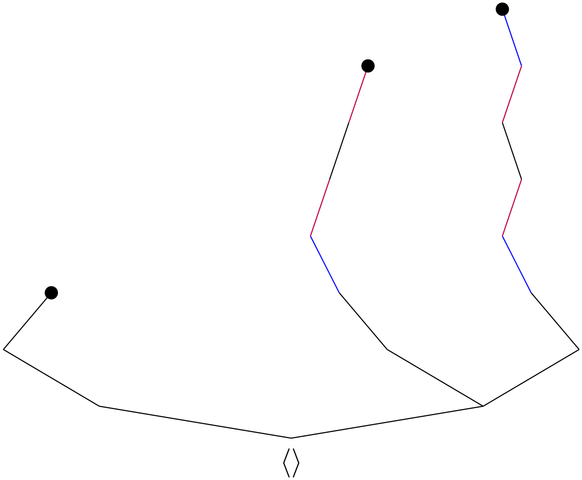
G a graph with three vertices and no edges

A tree A coding G



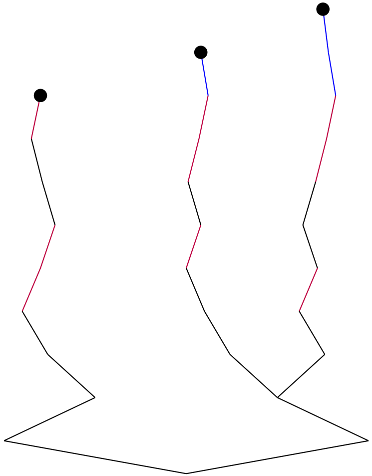
G a graph with three vertices and no edges

A tree B coding G. B is strictly similar to A.



Another tree C coding G

C is not strictly similar to A.



The triangle-free homogeneous graph has finite big Ramsey degrees

Theorem. (D.) For each finite triangle-free graph A, there is a positive integer $T(\mathrm{A}, \mathcal{K}_3)$ such that for any coloring of all copies of A in $\mathcal{H}_3$ into finitely many colors, there is a subgraph $\mathcal{H} \leq \mathcal{H}_3$, again universal triangle-free, such that all copies of A in $\mathcal{H}$ take no more than $T(\mathrm{A}, \mathcal{K}_3)$ colors.

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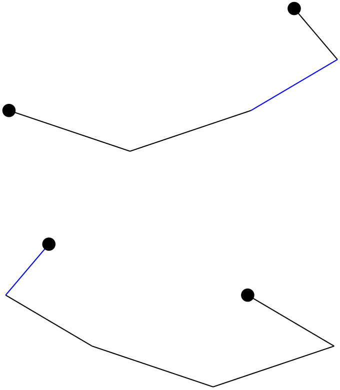
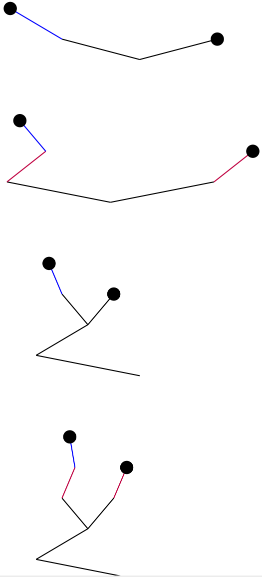
This is the first result on big Ramsey degrees of a homogeneous structure omitting a non-trivial substructure.

The two strict similarity types of Edge Codings



Non-edges have eight strict similarity types



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