Nonstandard integers
and Ramsey Theory of Diophantine equations

Mauro Di Nasso – Università di Pisa – Italia

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Introduction

In combinatorics of numbers one finds deep and fruitful interactions among diverse *non-elementary* methods, namely:

- Ergodic theory.
- Fourier analysis.
- Discrete topological dynamics.
- Algebra $(\beta\mathbb{N}, \oplus)$ in the space of ultrafilters on $\mathbb{N}$.

Recently, also nonstandard models of the integers and the techniques of nonstandard analysis have been applied to that area of research.

Areas of combinatorics where nonstandard methods have been applied:

- Additive combinatorics $\longrightarrow$ Density-dependent results for sets of integers (and generalizations to the context of amenable groups).
- Ramsey theory $\longrightarrow$ properties that are preserved under finite partitions.

The nonstandard natural numbers (hypernatural numbers) can play the role of ultrafilters on $\mathbb{N}$ and be used in Ramsey theory problems; in particular, they can be useful in the study of partition regularity of Diophantine equations.

Nonstandard Analysis, hyper-quickly

Nonstandard analysis essentially consists of two properties:

1. Every mathematical object of interest X is extended to an object ${}^*X$, called hyper-extension or nonstandard extension.
2. ${}^*X$ is a sort of weakly isomorphic copy of X, in the sense that it satisfies exactly the same elementary properties as X.

What do we mean by "elementary property"? This is made precise with the formal language of 1st order logic.

Transfer principle

If $P(A_1, \ldots, A_n)$ is any elementary property of $A_1, \ldots, A_n$ then

$$P(A_1, \ldots, A_n) \iff P({}^*A_1, \ldots, {}^*A_n)$$

Examples:

- The hyperintegers ${}^*\mathbb{Z}$ are a *discretely ordered ring*.
- The hyperreal numbers ${}^*\mathbb{R}$ are an *ordered field* that properly extends the real line $\mathbb{R}$.

$\mathbb{Z}$ and ${}^*\mathbb{Z}$, and similarly $\mathbb{R}$ and ${}^*\mathbb{R}$, cannot be distinguished by any elementary property.

- A property of X is *elementary* if it talks about elements of X ("first-order" property).
  E.g., the properties of ordered field are elementary properties of $\mathbb{R}$.

- A property of X is NOT *elementary* if it talks about subsets or functions of X ("second-order" property).
  E.g., the well-ordering property of $\mathbb{N}$ and the completeness property of $\mathbb{R}$ are not elementary.

Ultrapowers as nonstandard extensions

Nonstandard analysis is no more "exotic" than ultrafilters.

Indeed, nonstandard analysis can be seen as a general uniform framework where the ultraproduct construction is performed.

A typical model of nonstandard analysis is obtained by picking an ultrafilter $\mathcal{U}$ on a set of indexes I, and by letting the hyper-extensions be the corresponding ultrapowers:

$$^*X \;=\; X^I/\mathcal{U} \;=\; \mathrm{Fun}(I,X)/{\equiv_{\mathcal{U}}}$$

where $\equiv_{\mathcal{U}}$ is the equivalence relation:

$$f \equiv_{\mathcal{U}} g \iff \{i \mid f(i)=g(i)\} \in \mathcal{U}.$$

For instance, let $\mathcal{U}$ be an ultrafilter on I, and let

$$^*\mathbb{R} \ = \mathbb{R}^I/\mathcal{U} \ = \ \{[\sigma] \mid \sigma : I \to \mathbb{R}\}\,.$$

We can assume $\mathbb{R} \subseteq {}^*\mathbb{R}$ by identifying each $r \in \mathbb{R}$ with the equivalence class $[c_r]$ of the constant sequence with value r.

Relations and functions are naturally extended to ultrapowers. *E.g.*:

- $[\sigma] < [\tau] \Leftrightarrow \{i \mid \sigma(i) < \tau(i)\} \in \mathcal{U}$.
- $[\sigma] + [\tau] = [\vartheta]$ where $\vartheta(i) = \sigma(i) + \tau(i)$ for all i.

If $\sigma : I \to \mathbb{R}$ is not constant on any set in $\mathcal{U}$, then its equivalence class $[\sigma] \in {}^*\mathbb{R} \setminus \mathbb{R}$ is a new element.

The hyperreal numbers

As a proper extension of the real line, the hyperreal field ${}^*\mathbb{R}$ contains infinitesimal numbers $\varepsilon \neq 0$:

$$-\frac{1}{n} < \varepsilon < \frac{1}{n} \quad \text{for all } n \in \mathbb{N}$$

and infinite numbers Ω:

$$|\Omega| > n \quad \text{for all } n \in \mathbb{N}.$$

So, ${}^*\mathbb{R}$ is *not* Archimedean, and hence it is *not* complete (*e.g.*, the bounded set of infinitesimal numbers does not have a least upper bound).

Both the *Archimedean property* and the *completeness property* are not elementary properties of $\mathbb{R}$.

Standard Part

Every *finite* number $\xi \in {}^*\mathbb{R}$ has infinitesimal distance from a unique real number, called the standard part of ξ:

$$\xi \approx \mathrm{st}(\xi) \in \mathbb{R}$$

Proof. Let $\mathrm{st}(\xi) = \sup\{r \in \mathbb{R} \mid r < \xi\} = \inf\{r \in \mathbb{R} \mid r > \xi\}$.

So ${}^*\mathbb{R}$ consists of infinite numbers and of numbers of the form $r + \varepsilon$ where $r \in \mathbb{R}$ and $\varepsilon \approx 0$ is infinitesimal.

The hypernatural numbers

The hyperintegers ${}^*\mathbb{Z}$ are a *discretely ordered ring* whose positive part are the hypernatural numbers ${}^*\mathbb{N}$, which are a very special ordered semiring.

$$ {}^*\mathbb{N} \;=\; \{\underbrace{1, 2, \ldots, n, \ldots}_{\text{finite numbers}} \quad \underbrace{\ldots, N-2, N-1, N, N+1, N+2, \ldots}_{\text{infinite numbers}}\} $$

- Every $\xi \in {}^*\mathbb{R}$ has an *integer part*, *i.e.* there exists a unique hyperinteger $\nu \in {}^*\mathbb{Z}$ such that $\nu \leq \xi < \nu + 1$.

The hyperfinite sets

Fundamental objects are the hyperfinite sets, which retain all the elementary properties of finite sets.

Example:

For every $N < M$ in ${}^*\mathbb{N}$, the following interval is hyperfinite:

$$[N, M]_{{}^*\mathbb{N}} \ = \ \{\nu \in {}^*\mathbb{N} \mid N \leq \nu \leq M\}.$$

If $M - N$ is an infinite number then $[N, M]_{{}^*\mathbb{N}}$ is an infinite set.

In the ultrapower model, hyperfinite intervals correspond to ultraproducts of intervals $[n_i, m_i] \subset \mathbb{N}$ of unbounded length:

$$[N, M]_{{}^*\mathbb{N}} \ = \ \prod_{i \in I} [n_i, m_i]/\equiv_{\mathcal{U}} \ = \ \{[\sigma] \mid \sigma(i) \in [n_i, m_i] \text{ for every } i\}.$$

Why NSA in combinatorics?

- Arguments of elementary finite combinatorics can be used in a hyperfinite setting to prove results about infinite sets of integers, also in the case of null asymptotic density.

- Nonstandard proofs for density-depending results usually work also in the more general setting of amenable groups.

- The nonstandard integers (or hyperintegers) ${}^*\mathbb{Z}$ may serve as a sort of "bridge" between the *discrete* and the *continuum*.

- Tools from *analysis* and *measure theory*, such as Birkhoff Ergodic Theorem and Lebesgue Density Theorem, can be used in ${}^*\mathbb{Z}$.

- Hypernatural numbers can play the role of ultrafilters on $\mathbb{N}$ and be used in *Ramsey theory* problems (*e.g.*, partition regularity of Diophantine equations).

- Model-theoretic tools are available, most notably saturation. *E.g.*, saturation is needed for the *Loeb measure construction*.

Examples of nonstandard definitions

Definition

The upper asymptotic density of a set $A \subseteq \mathbb{N}$:

$$\overline{d}(A) \;=\; \limsup_{n\to\infty} \frac{|A \cap [1,n]|}{n}.$$

Definition (Nonstandard)

$\overline{d}(A) \geq \alpha$ if there exists an infinite $N \in {}^*\mathbb{N}$ such that

$$\frac{|{}^*A \cap [1,N]|}{N} \approx \alpha.$$

Definition

The upper Banach density of a set $A \subseteq \mathbb{Z}$:

$$\mathrm{BD}(A) = \lim_{n\to\infty}\left(\max_{k\in\mathbb{Z}} \frac{|A \cap [k+1, k+n]|}{n}\right)$$

Definition (Nonstandard)

$\mathrm{BD}(A) \geq \alpha$ if there exists an infinite interval I such that

$$\frac{|{}^*A \cap I|}{|I|} \approx \alpha.$$

Definition

A is thick if for every k there exists x such that $[x+1, x+k] \subseteq A$.

Definition (Nonstandard)

A is thick if $I \subset {}^*A$ for some infinite interval I.

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Definition

A is syndetic if there exists $k \in \mathbb{N}$ such that every interval $[x, x+k] \cap A \neq \emptyset$.

Definition (Nonstandard)

A is syndetic if ${}^*A$ has *finite gaps*,
i.e. if ${}^*A \cap I \neq \emptyset$ for every infinite interval I.

Definition

A is piecewise syndetic if $A = B \cap C$ with B *syndetic* and C *thick*.

Definition (Nonstandard)

A is piecewise syndetic if there exists an infinite interval I such that ${}^*A \cap I$ has *finite gaps*.

Nonstandard definitions usually simplify the formalism and make proofs more straight, since one avoids using sequences and the usual "ϵ-δ arguments".

Examples of nonstandard reasoning

Theorem.
The family of *piecewise syndetic* sets is *partition regular*.

Nonstandard proof: By induction, it is enough to check the property for 2-partitions $A = \mathrm{BLUE} \cup \mathrm{RED}$.

- Take hyper-extensions ${}^*A = {}^*\mathrm{BLUE} \cup {}^*\mathrm{RED}$, and pick an infinite interval I where ${}^*A$ has only finite gaps.
- If the *blue elements of ${}^*A$ have only finite gaps in I, then BLUE is piecewise syndetic.
- Otherwise, there exists an infinite interval $J \subseteq I$ without *blue elements, that is, J only contains *red elements of ${}^*A$. But then ${}^*\mathrm{RED}$ has only finite gaps in J, and hence RED is piecewise syndetic.

Here is a typical nonstandard argument about asymptotic densities:

- Suppose the Banach density $\mathrm{BD}(A) = \alpha > 0$.
- Take an infinite interval $I = [\Omega + 1, \Omega + N]$ of ${}^*\mathbb{Z}$ such that the relative density $|{}^*A \cap I|/N \approx \alpha$.
- Take the Loeb measure μ on I, that extends the "counting measure": for all *internal* $X \subseteq I$, it is $\mu(X) = \mathrm{st}(|X \cap I|/N)$.
- Consider the *shift operator* $T : \xi \mapsto \xi + 1$ (we agree that $T(\Omega + N) = \Omega + 1$).

- Apply Birkhoff Ergodic Theorem: For almost all $\xi \in I$ the following limit exists:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \chi_A(T^i(\xi))$$

- By the nonstandard characterization of Banach density, it is proved that such limits equal $\mathrm{BD}(A)$ for almost all $\xi \in I$:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \chi_A(T^i(\xi)) = \lim_{n\to\infty} \frac{|{}^*A \cap [\xi+1, \xi+n]|}{n} = \mathrm{BD}(A).$$

Let $A_\xi = ({}^*A - \xi) \cap \mathbb{N} = \{i \in \mathbb{N} \mid \xi + i \in {}^*A\}$. We have proved that for almost all ξ, the density $d(A_\xi) = \mathrm{BD}(A)$.

What does it mean?

Definition

$B \leq_{fe} A$: B is finitely embeddable in A if for every n, there exists a shift $x + (B \cap [1, n]) = A \cap [x + 1, x + n]$.

Finite embeddability preserves all finite configurations (*e.g.*, the existence of arbitrarily long arithmetic progressions).

- Fact: $B \leq_{fe} A$ if and only if there exists $\xi \in {}^*\mathbb{N}$ with $B = A_\xi$.

Theorem

Let $BD(A) = \alpha$. Then there exists sets $B \leq_{fe} A$ with asymptotic density $d(B) = \alpha$. (Actually, much more holds; e.g., one can assume Schnirelmann density $\sigma(B) = \alpha$.)

Hypernatural numbers as ultrafilters

In a nonstandard setting, every hypernatural number $\xi \in {}^*\mathbb{N}$ generates an ultrafilter on $\mathbb{N}$:

$$\mathfrak{U}_\xi = \{A \subseteq \mathbb{N} \mid \xi \in {}^*A\}$$

If we assume ${}^*\mathbb{N}$ to be $\mathfrak{c}^+$-saturated, then every ultrafilter on $\mathbb{N}$ is generated by some $\xi \in {}^*\mathbb{N}$ (actually, by at least $\mathfrak{c}^+$-many ξ).

In some sense, in a nonstandard setting every ultrafilter is a *principal* ultrafilter (it can be seen as the family of all properties satisfied by a single "ideal" element $\xi \in {}^*\mathbb{N}$).

u-equivalence

Definition

For $\xi, \zeta \in {}^*\mathbb{N}$, we say that $\xi \underset{u}{\sim} \zeta$ are u-equivalent if they generate the same ultrafilter $\mathfrak{U}_\xi = \mathfrak{U}_\zeta$.

So, $\xi \underset{u}{\sim} \zeta$ means that ξ and ζ are indistinguishable by any "standard property":

- For every $A \subseteq \mathbb{N}$ one has either $\xi, \zeta \in {}^*A$ or $\xi, \zeta \notin {}^*A$.

In *model-theoretic* terms, $\xi \underset{u}{\sim} \zeta$ means that $tp(\xi) = tp(\zeta)$ in the complete language containing a symbol for every relation.

${}^*\mathbb{N}$ as a topological space

There is a natural topology on ${}^*\mathbb{N}$, named the standard topology, whose basic (cl)open sets are the hyper-extensions: $\{{}^*A \mid A \subseteq \mathbb{N}\}$.

${}^*\mathbb{N}$ is compact but not Hausdorff; and in fact two elements ξ, ζ are *not* separated precisely when $\xi \underset{u}{\sim} \zeta$.

The Hausdorff quotient space ${}^*\mathbb{N}/\underset{u}{\sim}$ is isomorphic to $\beta\mathbb{N}$.

While the Stone-Čech compactification $\beta\mathbb{N}$ is the "largest" Hausdorff compactification of the discrete space $\mathbb{N}$, the hypernatural numbers ${}^*\mathbb{N}$ are a larger space with several nice properties.

1. ${}^*\mathbb{N}$ is compact and completely regular (but not Hausdorff).
[X is *completely regular* if for every closed C and $x \notin C$ there is a continuous $f : X \to \mathbb{R}$ with $f(x) = 0$ and $f \equiv 1$ on C.]

2. $\mathbb{N}$ is dense in ${}^*\mathbb{N}$.

3. Every $f : \mathbb{N} \to K$ where K is compact Hausdorff is naturally extended to a continuous $\overline{f} : {}^*\mathbb{N} \to K$ by letting $\overline{f}(\xi)$ be the unique $x \in K$ that is "near" to ${}^*f(\xi)$, in the sense that ${}^*f(\xi) \in {}^*U$ for all neighborhoods U of x.

4. By means of hyper-extensions, every function $f : \mathbb{N}^k \to \mathbb{N}$ is extended to a continuous ${}^*f : {}^*\mathbb{N}^k \to {}^*\mathbb{N}$ that satisfies the same "elementary properties" as f. In particular, sum and product on $\mathbb{N}$ are extended to commutative operations on ${}^*\mathbb{N}$.

Algebra on ultrafilters

The space of ultrafilters $\beta\mathbb{N}$ has a natural pseudosum operation $\oplus$ that extends addition on $\mathbb{N}$ and makes $(\beta\mathbb{N}, \oplus)$ a right topological semigroup:

$$A \in \mathcal{U} \oplus \mathcal{V} \iff \{n \mid A - n \in \mathcal{V}\} \in \mathcal{U}$$

where $A - n = \{m \mid m + n \in A\}$. (Similarly with multiplication.)

How is the pseudo-sum $\oplus$ in $\beta\mathbb{N}$ related to the sum $+$ in ${}^*\mathbb{N}$?

Caution! In general, $\mathfrak{U}_\xi \oplus \mathfrak{U}_\zeta \neq \mathfrak{U}_{\xi+\zeta}$.

In fact, while $({}^*\mathbb{N}, +)$ is the positive part of an ordered ring, $(\beta\mathbb{N}, \oplus)$ is just a semiring whose center is $\mathbb{N}$.

Definition (Nonstandard)

$A \in \mathfrak{U}_\xi \oplus \mathfrak{U}_\zeta \iff \xi \in {}^*A_\zeta$ where $A_\zeta = \{n \in \mathbb{N} \mid \zeta + n \in {}^*A\}$.

Theorem

Assume $a_{k+1} - a_k \nearrow \infty$ and let $A = \bigcup_n [a_{2n}, a_{2n+1})$. For every non-principal ultrafilter $\mathcal{U}$ there exists a non-principal ultrafilter $\mathcal{V}$ such that $A \in \mathcal{U} \oplus \mathcal{V} \Leftrightarrow A \notin \mathcal{V} \oplus \mathcal{U}$. So, the center of $(\beta\mathbb{N}, \oplus)$ is $\mathbb{N}$.

Nonstandard proof.

Pick an infinite ξ such that $\mathcal{U} = \mathfrak{U}_\xi$, and let $\nu \in {}^*\mathbb{N}$ be such that $\xi \in [a_\nu, a_{\nu+1})$. Assume that ν is even, so that $\xi \in {}^*A$ (the case ν odd is entirely similar). We distinguish two cases.

1. If $a_{\nu+1} - \xi$ is infinite, let $\mathcal{V} = \mathfrak{U}_{a_{\nu+1}}$. Then $A \in \mathcal{U} \oplus \mathcal{V}$ because $\xi + n \in [a_\nu, a_{\nu+1}) \subset {}^*A$ for all n, and so trivially $a_{\nu+1} \in {}^*A_\xi = {}^*\mathbb{N}$. Besides, $A \notin \mathcal{V} \oplus \mathcal{U}$ because $a_{\nu+1} + n \notin {}^*A$ for every n, and so trivially $\xi \notin {}^*A_{a_{\nu+1}} = {}^*\emptyset = \emptyset$.

2. If $a_{\nu+1} - \xi$ is finite, let $\mathcal{V} = \mathfrak{U}_{a_\nu}$. Then $A \notin \mathcal{U} \oplus \mathcal{V}$ because $A_\xi = \{n \mid \xi + n \in {}^*A\}$ is finite, and so $a_\nu \notin {}^*A_\xi$.
Besides, $A \in \mathcal{V} \oplus \mathcal{U}$ because $a_\nu + n \in [a_\nu, a_{\nu+1}) \subset {}^*A$ for all n, and so trivially $\xi \in {}^*A_{a_\nu} = {}^*\mathbb{N}$. □

Topological dynamics in ${}^*\mathbb{N}$

Let us consider the shift operator S on the compact space ${}^*\mathbb{N}$:

$$S : \nu \longmapsto \nu + 1$$

1. A point $\nu \in {}^*\mathbb{N}$ is recurrent if and only if $\mathfrak{U}_\nu = \mathfrak{U}_\mu \oplus \mathfrak{U}_\nu$ for some $\mu \in {}^*\mathbb{N}$.
2. A point $\nu \in {}^*\mathbb{N}$ is uniformly recurrent if and only if the ultrafilter $\mathfrak{U}_\nu$ is minimal. So, if $\nu \in {}^*A$ then A contains arbitrarily long arithmetic progressions.
3. A point ν generates an idempotent ultrafilter $\mathfrak{U}_\nu = \mathfrak{U}_\nu \oplus \mathfrak{U}_\nu$ if and only if ν is "self-recurrent" in the following sense:

$$\nu \in {}^*A \implies \nu + a \in {}^*A \text{ for some } a \in A.$$

Iterated hyper-extensions of $\mathbb{N}$

By iterating hyper-extensions, one obtains the *hyper-hypernatural numbers* ${}^{**}\mathbb{N}$, the *hyper-hyper-hypernatural numbers* ${}^{***}\mathbb{N}$, and so forth.

- The natural numbers are an initial segment of the hypernatural numbers: $\mathbb{N} < {}^*\mathbb{N} \setminus \mathbb{N}$.
- By *transfer*, ${}^*\mathbb{N} < {}^{**}\mathbb{N} \setminus {}^*\mathbb{N}$.
- If $\nu \in {}^*\mathbb{N} \setminus \mathbb{N}$ then ${}^*\nu \in {}^{**}\mathbb{N} \setminus {}^*\mathbb{N}$, and so ${}^*\nu > \mu$ for all $\mu \in {}^*\mathbb{N}$.
- If $\Omega \in {}^{**}\mathbb{N}$, one defines $\mathfrak{U}_\Omega = \{A \subseteq \mathbb{N} \mid \Omega \in {}^{**}A\}$. Since $\nu \in {}^*A \Leftrightarrow {}^*\nu \in {}^{**}A$, one has $\mathfrak{U}_{{}^*\nu} = \mathfrak{U}_\nu$, that is, ${}^*\nu \underset{u}{\sim} \nu$.

Recall that in general $\mathfrak{U}_\nu \oplus \mathfrak{U}_\mu \neq \mathfrak{U}_{\nu+\mu}$. However, by using iterated hyper-estensions, one has a nice nonstandard characterization of pseudo=sums:

- $\mathfrak{U}_\nu \oplus \mathfrak{U}_\mu \ = \ \mathfrak{U}_{\nu + {}^*\mu}$.
- $\mathfrak{U}_\nu \oplus \mathfrak{U}_\mu \oplus \mathfrak{U}_\vartheta \ = \ \mathfrak{U}_{\nu + {}^*\mu + {}^{**}\vartheta}$; and so forth.

Idempotent points

$\mathfrak{U}_\nu = \mathfrak{U}_\nu \oplus \mathfrak{U}_\nu$ is idempotent if and only if $\nu + {}^*\nu \underset{u}{\sim} \nu$

This characterization makes it easier to handle idempotent ultrafilters and their combinations.

Let us see one simple example.

Theorem (Bergelson-Hindman 1990)

Let $\mathcal{U}$ be an idempotent ultrafilter. Then every $A \in 2\mathcal{U} \oplus \mathcal{U}$ contains an arithmetic progression of length 3.

Nonstandard proof.

Let ν be such that $\mathcal{U} = \mathfrak{U}_\nu$, so $\nu \underset{u}{\sim} \nu + {}^*\nu$. Then

- $\xi = 2\nu + {}^{**}\nu$
- $\zeta = 2\nu + {}^*\nu + {}^{**}\nu$
- $\vartheta = 2\nu + 2{}^*\nu + {}^{**}\nu$

are *u*-equivalent numbers of ${}^{***}\mathbb{N}$ that generate $\mathcal{V} = 2\mathcal{U} \oplus \mathcal{U}$. For every $A \in \mathcal{V}$, the elements $\xi, \zeta, \vartheta \in {}^{***}A$ form a 3-term arithmetic progression and so, by *transfer*, there exists a 3-term arithmetic progression in A.

Idempotent ultrafilters and Rado's Theorem

The previous argument can be generalized, and one can prove the following ultrafilter version of Rado's theorem.

Theorem ("Idempotent Ultrafilter Rado" - DN 2015)

Let $c_1X_1 + \ldots + c_nX_n = 0$ be a Diophantine equation with $n \geq 3$. If $c_1 + \ldots + c_n = 0$ then there exist $a_1, \ldots, a_{n-1} \in \mathbb{N}$ such that for every idempotent ultrafilter $\mathcal{U}$, the ultrafilter

$$\mathcal{V} = a_1\mathcal{U} \oplus \ldots \oplus a_{n-1}\mathcal{U}$$

is an injective PR-witness, i.e. for every $A \in \mathcal{V}$ there exist distinct $x_1, \ldots, x_n \in A$ with $c_1x_1 + \ldots + c_kx_k = 0$.

Let $\mathcal{U} = \mathfrak{U}_\nu$ be any idempotent ultrafilter, where $\nu \in {}^*\mathbb{N}$.

Let us denote by $u_1 = \nu \in {}^*\mathbb{N}$, $u_2 = {}^*\nu \in {}^{**}\mathbb{N}$, $u_3 = {}^{**}\nu \in {}^{***}\mathbb{N}$, and so forth.

Let $a_1, \ldots, a_{n-1}$ be arbitrary integers, and consider the following elements in ${}^{n*}\mathbb{N}$, the n-th iterated hyper-extension of $\mathbb{N}$:

$$\begin{array}{lllllllllllllll}
\zeta_1 & = & a_1 u_1 & + & a_1 u_2 & + & a_2 u_3 & + & a_3 u_4 & + & \ldots & + & a_{n-2} u_{n-1} & + & a_{n-1} u_n \\
\zeta_2 & = & a_1 u_1 & + & 0 & + & a_2 u_3 & + & a_3 u_4 & + & \ldots & + & a_{n-2} u_{n-1} & + & a_{n-1} u_n \\
\zeta_3 & = & a_1 u_1 & + & a_2 u_2 & + & 0 & + & a_3 u_4 & + & \ldots & + & a_{n-2} u_{n-1} & + & a_{n-1} u_n \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
\zeta_{n-1} & = & a_1 u_1 & + & a_2 u_2 & + & a_3 u_3 & + & \ldots & + & a_{n-2} u_{n-2} & + & 0 & + & a_{n-1} u_n \\
\zeta_n & = & a_1 u_1 & + & a_2 u_2 & + & a_3 u_3 & + & \ldots & + & a_{n-2} u_{n-2} & + & a_{n-1} u_{n-1} & + & a_{n-1} u_n
\end{array}$$

Then $\zeta_1 \underset{u}{\sim} \zeta_2 \underset{u}{\sim} \ldots \underset{u}{\sim} \zeta_n$ generate the same ultrafilter, namely:

$$\mathcal{V} \ = \ a_1\mathcal{U} \oplus a_2\mathcal{U} \oplus \ldots \oplus a_{n-1}\mathcal{U}$$

Now, $c_1\zeta_1 + \ldots + c_n\zeta_n = 0$ if and only if the coefficients a_i fulfill the following conditions:

$$\begin{cases} (c_1 + c_2 + \ldots + c_n) \cdot a_1 = 0 \\ c_1 \cdot a_1 + (c_3 + \ldots + c_n) \cdot a_2 = 0 \\ (c_1 + c_2) \cdot a_2 + (c_4 + \ldots + c_n) \cdot a_3 = 0 \\ \qquad\qquad \vdots \\ (c_1 + c_2 + \ldots + c_{n-3}) \cdot a_{n-3} + (c_{n-1} + c_n) \cdot a_{n-2} = 0 \\ (c_1 + c_2 + \ldots + c_{n-2}) \cdot a_{n-2} + c_n \cdot a_{n-1} = 0 \\ (c_1 + c_2 + \ldots + c_n) \cdot a_{n-1} = 0 \end{cases}$$

The first and the last equations are trivially satisfied because of the hypothesis $c_1 + c_2 + \ldots + c_n = 0$.

The remaining $n-2$ equations are satisfied by (infinitely many) $a_1, \ldots, a_{n-1} \in \mathbb{N}$, that can be explicitly given in terms of the c_i.

Since all the $a_i \neq 0$, the numbers ζ_is are mutually distinct and we can apply the nonstandard characterization of injective PR.

PR of diophantine equations

Definition

An equation $F(x_1, \dots, x_n) = 0$ is partition regular on $\mathbb{N}$ if for every finite coloring of $\mathbb{N}$ there exist a monochromatic solution, *i.e.* monochromatic elements $a_1, \dots, a_n$ such that $F(a_1, \dots, a_n) = 0$.

- By *Schur's Theorem*, the equation $X + Y = Z$ is PR.
- By *van der Waerden's theorem*, the equation $X + Y = 2Z$ is PR. (Solutions are the 3-term arithmetic progressions.)
- However, the equation $X + Y = 3Z$ is *not* PR!

The problem of partition regularity of linear diophantine equations was completely solved by Richard Rado.

Theorem (Rado 1933)

The diophantine equation $c_1X_1 + \ldots + c_nX_n = 0$ is PR if and only if $\sum_{i \in I} c_i = 0$ for some (nonempty) $I \subseteq \{1, \ldots, k\}$.

Numerous PR results have been proved for linear equations (especially about infinite systems), but the study on the nonlinear case has been sporadic, until very recently.

Nonstandard characterization

Definition

An equation $F(x_1, \dots, x_n) = 0$ is partition regular on $\mathbb{N}$ if for every finite coloring of $\mathbb{N}$ there exist monochromatic elements $a_1, \dots, a_n$ such that $F(a_1, \dots, a_n) = 0$.

Definition (Nonstandard)

An equation $F(x_1, \dots, x_n) = 0$ is partition regular on $\mathbb{N}$ if there exist $\xi_1 \underset{u}{\sim} \dots \underset{u}{\sim} \xi_n$ in ${}^*\mathbb{N}$ such that ${}^*F(\xi_1, \dots, \xi_n) = 0$.

So, Schur's Theorem states the existence of hypernatural numbers:

$$\xi \underset{u}{\sim} \zeta \underset{u}{\sim} \xi + \zeta$$

Properties of u-equivalence

Theorem

(1) *If* $\xi \not\sim_u \zeta$ *then* $|\xi - \zeta|$ *is infinite.*

Let $f : \mathbb{N} \to \mathbb{N}$ *be any function. Then*

(2) *If* $\xi \sim_u \zeta$ *then* ${}^*f(\xi) \sim_u {}^*f(\zeta)$.

(3) *If* ${}^*f(\xi) \sim \xi$ *then* ${}^*f(\xi) = \xi$.

The last property corresponds to the following basic (non-trivial) fact on ultrafilters:

$$f(\mathcal{U}) = \mathcal{U} \implies \{n \mid f(n) = n\} \in \mathcal{U}$$

The nonstandard framework of hypernatural numbers and u-equivalence relation, combined with the use of special ultrafilters, revealed useful to the study the *nonlinear* case.

Useful observation

If $F(x_1, \ldots, x_n) = 0$ is a homogeneous PR equation, then there exist a multiplicatively idempotent ultrafilter $\mathcal{U}$ which is a witness of the PR.

Proof. The set of all witnesses $\mathcal{U}$ is a closed bilateral ideal of $(\beta\mathbb{N}, \odot)$.

Examples of nonlinear equations

Theorem (Hindman 2011)

Equations $X_1 + X_2 + \ldots + X_n = Y_1 \cdot Y_2 \cdot \cdots \cdot Y_m$ *are PR.*

Nonstandard proof.
E.g., let us consider $x_1 + x_2 = y_1 \cdot y_2 \cdot y_3$. The linear equation $x_1 + x_2 = y$ is PR, and so there exist $\alpha_1 \underset{u}{\sim} \alpha_2 \underset{u}{\sim} \beta$ such that $\alpha_1 + \alpha_2 = \beta$. We can assume that $\beta \underset{u}{\sim} \beta\,{}^*\beta$ is *multiplicatively idempotent*. Then

- $\gamma_1 = \alpha_1{}^*\beta\,{}^{**}\beta = \alpha_1{}^*(\beta\,{}^*\beta) \underset{u}{\sim} \alpha_1{}^*\beta \underset{u}{\sim} \beta$
- $\gamma_2 = \alpha_2{}^*\beta\,{}^{**}\beta \underset{u}{\sim} \beta$

are such that

$$\gamma_1 + \gamma_2 = (\alpha_1 + \alpha_2)\,{}^*\beta\,{}^{**}\beta = \beta\,{}^*\beta\,{}^{**}\beta$$

By generalizing the previous nonstandard argument, the following generalization of Hindman's Theorem is proved:

Theorem (Luperi Baglini 2013)

Let $a_1X_1 + \ldots + a_nX_n = 0$ be partition regular. Then for every choice of finite sets $F_1, \ldots, F_n \subseteq \{1, \ldots, m\}$, the following polynomial equation is partition regular:
(Variables X_i and Y_j must be distinct.)

$$a_1X_1\left(\prod_{j\in F_1} Y_j\right) + a_2X_2\left(\prod_{j\in F_2} Y_j\right) + \ldots + a_nX_n\left(\prod_{j\in F_n} Y_j\right) = 0.$$

Hindman's theorem is the case where one considers the equation $X_1 + X_2 + \ldots + X_n - Y_1 = 0$, and finite sets $F_1 = F_2 = \ldots = F_n = \emptyset$, $F_{n+1} = \{2, \ldots, m\}$.

Examples of nonlinear equations

Theorem

The equation $X^2 + Y^2 = Z$ is not partition regular.

Proof. By contradiction, let $\alpha \underset{u}{\sim} \beta \underset{u}{\sim} \gamma$ be such that $\alpha^2 + \beta^2 = \gamma$. α, β, γ are even numbers, since they cannot all be odd; then

$$\alpha = 2^a \alpha_1, \quad \beta = 2^b \beta_1, \quad \gamma = 2^c \gamma_1$$

where $a \underset{u}{\sim} b \underset{u}{\sim} c$ are positive and $\alpha_1 \underset{u}{\sim} \beta_1 \underset{u}{\sim} \gamma_1$ are odd.

Case 1: If $a < b$ then $2^{2a}(\alpha_1^2 + 2^{2b-2a}\beta_1^2) = 2^c \gamma_1$. Since $\alpha_1^2 + 2^{2b-2a}\beta_1^2$ and γ_1 are odd, it follows that $2a = c \underset{u}{\sim} a$. But then $2a = a$ and so $a = 0$, a contradiction. (Same proof if $b > a$.)

Case 2: If $a = b$ then $2^{2a}(\alpha_1^2 + \beta_1^2) = 2^c \gamma_1$. Since α_1, β_1 are odd, $\alpha_1^2 + \beta_1^2 \equiv 2 \mod 4$, and so $2^c \gamma_1 = 2^{2a+1}\alpha_2$ where α_2 is odd. But then $2a + 1 = c \underset{u}{\sim} a$ and so $2a + 1 = a$, a contradiction.

Some new results

By exploiting the properties of *u*-equivalence in ${}^*\mathbb{N}$, we isolated a large class of *non* PR equations (joint work with M. Riggio).

Theorem (DN-Riggio 2016)

Every Fermat-like equation $x^n + y^m = z^k$ where $k \notin \{n, m\}$ is not partition regular.

Grounding on combinatorial properties of positive density sets and IP sets, and exploiting the algebraic structure of $(\beta\mathbb{N}, \oplus, \odot)$, several positive results are proved.

A really useful fact:

Joint PR Lemma (DN-Luperi Baglini 2016)

Assume that the same ultrafilter $\mathcal{U}$ is a PR-witness of equations $f_i(x_{i,1},\ldots,x_{i,n_i})=0$, where f_i have pairwise disjoint sets of variables. Then $\mathcal{U}$ is also a PR-witness of the following system:

$$\begin{cases} f_i(x_{i,1},\ldots,x_{i,n_i})=0 \quad i=1,\ldots,k; \\ x_{1,1}=\ldots=x_{k,1}. \end{cases}$$

Example

If $\mathcal{U}$ is a PR-witness of $u - v = t^2$, then $\mathcal{U}$ is also a PR-witness of the system:

$$\begin{cases} u_1 - y = x^2; \\ u_2 - z = y^2. \end{cases}$$

So, configuration $\{x, y, z, y + x^2, z + y^2\}$ is PR. (This fact was proved by Bergelson-Johnson-Moreira 2015.)

Many other similar examples are easily found.

Theorem (DN-Luperi Baglini)

The PR of every Diophantine equation

$$a_1X_1 + \ldots + a_kX_k = P(Y_1, \ldots, Y_n)$$

where the polynomial P has no constant term and the Rado's condition holds in the linear part, is witnessed by every ultrafilter

$$\mathcal{U} \in \overline{K(\odot)} \cap \overline{\mathbb{I}(\oplus)} \cap \mathcal{BD}.$$

"Rado's condition" means that $\sum_{i \in I} a_i = 0$ for some nonempty $I \subseteq \{1, \ldots, k\}$.

A key ingredient in the proof is a result by Bergelson-Furstenberg-McCutcheon of 1996.

By applying the Joint PR Lemma, we obtain:

Corollary (Bergelson-Johnson-Moreira 2015)

For all polynomials $P_i(y_i) \in \mathbb{Z}[y_i]$ with $P_i(0) = 0$, every $\mathcal{U} \in \overline{K(\odot)} \cap \overline{\mathbb{I}(\oplus)} \cap \mathcal{BD}$ is a PR-witness of the system:

$$\begin{cases} x_1 - y_1 = P_1(y_0); \\ x_2 - y_2 = P_2(y_1); \\ \dots \\ x_k - y_k = P_k(y_{k-1}). \end{cases}$$

With the help of u-equivalence one proves necessary conditions.

Theorem (DN-Luperi Baglini)

If a Diophantine equation of the form

$$P_1(x_1) + \ldots + P_k(x_k) = 0$$

where P_i has no constant terms is PR then the following "Rado's condition" is satisfied:

- *There exists a nonempty $I \subseteq \{1, \ldots, k\}$ such that*
  - $\deg P_i = \deg P_j$ *for all* $i, j \in I$;
  - $\sum_{i \in I} c_i = 0$ *where c_i is the leading term of P_i.*

By combining, we obtain a full characterization for a large class of equations.

Theorem (DN-Luperi Baglini)

A Diophantine equation of the form

$$a_1 X_1 + \ldots + a_k X_k = P(Y)$$

where the nonlinear polynomial P has no constant term is PR if and only if "Rado's condition" holds in the linear part, i.e. $\sum_{i \in I} a_i = 0$ *for some nonempty* $I \subseteq \{1, \ldots, k\}$.

Another general consequence:

Corollary

Every Diophantine equation of the form

$$a_1X_1^k + \ldots + a_nX_n^k = P_1(Y_1) + \ldots + P_h(Y_h)$$

where the polynomials P_j have pairwise different degrees $\neq k$ and no constant term, and where $\sum_{i\in I} a_i \neq 0$ for every (nonempty) $I \subseteq \{1,\ldots,k\}$, is not PR.

Some examples

- Khalfah and E. Szemerédi (2006) proved that if $P(Z) \in \mathbb{Z}[Z]$ takes even values on some integer, then in every finite coloring $X + Y = P(Z)$ has a solution with X, Y monochromatic. However, by our result, $X + Y = P(Z)$ is not PR for any nonlinear P.

- $X - 2Y = Z^2$ is not PR (while $X - Y = Z^2$ is).
  (This problem was posed by V. Bergelson in 1996.)

- Equation $X_1 - 2X_2 + X_3 = Y^k$ are PR. So, in every finite coloring of the natural numbers one finds monochromatic configurations of the form $\{a, b, c, 2a - b + c^k\}$; *etc.*

Some more examples

- Configuration $\{a, b, c, a + b, a \cdot c\}$ is PR;
- Configuration $\{a, b, c, d, a + b, c + d, (a + b) \cdot (c + d)\}$ is PR;
- Configuration $\{a, b, c, a - 17b, (a - 17b) \cdot c\}$ is PR;
- $\ldots$

OPEN QUESTION

Is the Pythagorean equation partition regular?

$$X^2 + Y^2 = Z^2$$

- $X^2 + Y^2 = Z^2$ is PR for 2-colorings (computer-assisted proof by Heule - Kullmann - Marek 2016).
- $X + Y = Z$ is PR (Schur's Theorem).
- $X^2 + Y = Z$ is PR (corollary of Sarkozy - Fürstenberg 1978).
- $X + Y = Z^2$ is *not* PR (Csikvári - Gyarmati - Sárközy 2012).
- $X^2 + Y^2 = Z$ is *not* PR (DN - Riggio 2016).
- $X^2 + Y = Z^2$ is PR (Moreira 2016)
- $X_1X_2 + Y^2 = Z_1Z_2$ is PR (DN - Luperi Baglini 2016).
- $X^n + Y^n = Z^k$ where $k \neq n$ are not PR (DN - Riggio 2016).
- $X^n + Y^n = Z^n$ where $n > 2$ has no solutions (Fermat's Thm!).

Ramsey Theory of Equations and related topics

16-17 Feb 2018 Pisa (Italy)

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AIMS

In 2016 two long standing open problems in Ramsey Theory of equations and polynomial configurations have been solved: the Boolean Pythagorean triples problem and the partition regularity of the configuration {x,x+y,xy}.
For this reason, problems regarding the partition regularity of nonlinear Diophantine equations and polynomial configurations are now in the spotlight of the mathematical community. An interesting feature of this topic is that different non-elementary techniques, including ultrafilters, ergodic theory, nonstandard analysis, semigroup theory and topological dynamics, can be applied to attack problems.
The aim of this Workshop is to present aspects of several of these different techniques, as well as to discuss some interesting related problems.

LIST OF INVITED SPEAKERS

Ben Barber (University of Bristol);
Lorenzo Carlucci (University of Rome I "La Sapienza");
Alexander Fish (University of Sidney);
David Gunderson (University of Manitoba);
Oliver Kullmann (Swansea University);
Hanno Lefmann (Chemnitz University of Technology);
Sofia Lindqvist (University of Oxford);
Sean Prendiville (University of Manchester);
Dona Strauss (University of Leeds);
Luca Q. Zamboni (Université Lyon 1).

“Nonstandard” references

- DN - Goldbring - Lupini, *Nonstandard Methods in Ramsey Theory and Combinatorial Number Theory*, book in preparation.
- DN, Ultrafilters as hypernatural numbers, chapter in *“Nonstandard Analysis for the Working Mathematician”* 2nd edition (Loeb and Wolff, eds.), Springer, 2015.
- DN, Iterated hyper-extensions and an idempotent ultrafilter proof of Rado’s theorem, *PAMS*, 2015.
- DN, A taste of nonstandard methods in combinatorics of numbers, in *Geometry, Structure and Randomness in Combinatorics* (Matoušek, Nešetřil and Pellegrini, eds.), 2015.
- DN - Riggio, Fermat-like equations that are not partition regular, *Combinatorica*, to appear.
- DN - Luperi Baglini, Ramsey properties of nonlinear diophantine equations, *Adv. Math.*, to appear.
- Luperi Baglini, Hyperintegers and Nonstandard Techniques in Combinatorics of Numbers, Ph.D. Thesis, Università di Siena, 2012.
- Luperi Baglini, Partition regularity of nonlinear polynomials: a nonstandard approach, *Integers*, 2014.
- Luperi Baglini, A nonstandard technique in combinatorial number theory, *European J. Combin.*, 2015.


The End

Ramsey Theorem

Let us see how iterated hyper-extensions can be used in Ramsey Theory.

Theorem (Ramsey – Infinite version)

Let $[\mathbb{N}]^k = \{\{n_1 < \ldots < n_k\} \mid n_s \in \mathbb{N}\} = C_1 \cup \ldots \cup C_r$ *be a finite coloring of the k-tuples. Then there exists an infinite homogeneous set* H*, i.e. all k-tuples from* H *are monochromatic:* $[H]^k \subseteq C_i$.

Let us give a proof in the hypernatural setting.

$\mathbf{k = 1}$. Pick an infinite $\xi \in {}^*\mathbb{N}$. Since ${}^*\mathbb{N} = {}^*C_1 \cup \ldots \cup {}^*C_r$ there exists a color i such that $\xi \in {}^*C_i$. But ξ is infinite, so C_i must be an infinite set.

Ramsey Theorem by iterated hyper-extensions

$\mathbf{k} = \mathbf{2}$. Let a finite coloring $[{}^{**}\mathbb{N}]^2 = {}^{**}C_1 \cup \ldots \cup {}^{**}C_r$ be given.

Pick an infinite $\nu \in {}^*\mathbb{N}$. Then $\{\nu, {}^*\nu\} \in {}^{**}C_i$ for some i.

We will prove that there exists an infinite $X = \{x_1 < x_2 < \ldots\}$ such that $\{x_s, x_t\} \in C_i$ for all $s < t$.

$\nu \in \{x \in {}^*\mathbb{N} \mid \{x, {}^*\nu\} \in {}^{**}C_i\} = {}^*\{x \in \mathbb{N} \mid \{x, \nu\} \in {}^*C_i\} = {}^*A$.

Pick $x_1 \in A$, so $\{x_1, \nu\} \in {}^*C_i$.

Then $\nu \in {}^*\{x \in \mathbb{N} \mid \{x_1, x\} \in C_i\} = {}^*B_1$.

$\nu \in {}^*A \cap {}^*B_1 \Rightarrow A \cap B_1$ is infinite: pick $x_2 \in A \cap B_1$ with $x_2 > x_1$.

$x_2 \in B_1 \Rightarrow \{x_1, x_2\} \in C_i$.

$x_2 \in A \Rightarrow \{x_2, \nu\} \in {}^*C_i \Rightarrow \nu \in {}^*\{x \in \mathbb{N} \mid \{x_2, x\} \in {}^*C_1\} = {}^*B_2$.

$\nu \in {}^*A \cap {}^*B_1 \cap {}^*B_2 \Rightarrow$ we can pick $x_3 \in A \cap B_1 \cap B_2$ with $x_3 > x_2$.

$x_3 \in B_1 \cap B_2 \Rightarrow \{x_1, x_3\}, \{x_2, x_3\} \in C_i$, and so forth.

The infinite set $H = \{a_n \mid n \in \mathbb{N}\}$ is such that $[H]^2 \subset C_i$.